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On Lorentzian Ricci-Flat Homogeneous Manifolds

by

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Contents

Introduction

A homogeneous space is the coset manifold G/H, where G is a Lie group, and H is a closed Lie subgroup of G. The canonic mappings related to a homogeneous space are the projection, and the left-action. The differential of the left-action extends tensors of special form to invariant tensor fields on the manifold G/H. For instance, an invariant metric on G/H originates from a single scalar product B.

A homogeneous triple associated to the homogeneous space G/H with invariant metric is $(\mathfrak{g}, \mathfrak{h}, B)$, where $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{h} \subset \mathfrak{g}$ is the subalgebra induced by the subgroup H, and B is the scalar product on a vector space complement $\mathfrak{m}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ induced by the metric.

For a homogeneous space with invariant metric, geometric notions such as the Levi-Civita connection, and the curvature can be derived at a single point of G/H in terms of the homogeneous triple. The curvature tensor obtained this way translates to the invariant curvature tensor field on the semi-Riemannian manifold G/H. Locally, the homogeneous triple uniquely determines the corresponding semi-Riemannian homogeneous space.

In this thesis, we are interested in homogeneous triples that relate to homogeneous spaces with the following geometric properties:

$$\left.\begin{array}{l}\text{The invariant metric is Lorentzian.}\\ \text{The Riemannian curvature is non-zero.}\\ \text{The space is Ricci-flat.}\end{array}\right\} \qquad (0.1)$$

Metrics of index 1 are the core in relativity theory. From the geometric viewpoint, Riemannian-flat homogeneous spaces are not particularly interesting. A Ricci-flat homogeneous space is naturally an Einstein manifold.

In [CW70], M. Cahen and N. Wallach classify the symmetric triples of geometry (0.1). In [FM05], J. Figueroa-O'Farrill et al. construct further examples of such homogeneous triples. To the best of our knowledge, Lie groups with left-invariant metric and geometry (0.1) have not been explicitly mentioned in the literature.

A homogeneous triple is the combination of a homogeneous pair $(\mathfrak{g}, \mathfrak{h})$ with an invariant scalar product B on a vector space complement $\mathfrak{m}$. In [Ko95], B. Komrakov gives a computer generated classification of 4-dimensional homogeneous pairs with $\dim \mathfrak{h} \geq 1$. We assume that his classification is correct. Then, any 4-dimensional homogeneous triple with $\dim \mathfrak{h} \geq 1$ is the combination of a homogeneous pair in Komrakov's list with an invariant scalar product. In order to determine all 4-dimensional homogeneous triples with $\dim \mathfrak{h} \geq 1$ and geometry (0.1), we simply investigate the geometry of all possible combinations.

Although Komrakov's list of pairwise non-isomorphic homogeneous pairs extends over more than 80 pages, this strategy is doable. In [Ko01], B. Komrakov provides all invariant Lorentzian scalar products to each of the 4-dimensional homogeneous pairs. The efforts are worthwhile: We detect homogeneous triples, which are of the desired geometry and have not been covered by previous work.

The results lead us to introduce a new class of Lorentzian homogeneous triples $\mathcal{H}_m^n$. To the best of our knowledge, $\mathcal{H}_m^n$ includes all homogeneous triples with geometry (0.1) that are mentioned in the literature. In addition, the class covers all homogeneous triples of this particular geometry we could detect.

Our construction is an essential extension to previous work. The elements in $\mathcal{H}_m^n$ do not align in a vector space, but the parameters are subject to non-linear equations. Furthermore, $\mathcal{H}_m^0$ corresponds to Lie groups with left-invariant Lorentzian metric, which are Ricci-flat, but not Riemannian-flat for a certain choice of parameters.

We structure the thesis in the following way. In the introductory chapter, we cover the mathematical concepts that are relevant to our work. Our emphasis is on the geometric properties of a homogeneous space with invariant metric that can be derived in terms of an associated homogeneous triple.

In the second chapter, we give a concise overview on previous work that was useful to us. For instance, the classification of low-dimensional semi-Riemannian homogeneous spaces by B. Doubrov and B. Komrakov is an excellent reference to get acquainted with the subject. However, to several related publications we point later on.

In Chapter 3, we process the computer generated list of 4-dimensional homogeneous pairs with $\dim \mathfrak{h} \geq 1$ stated in [Ko01]. We argue that 10 homogeneous *pairs* in Komrakov's list extend to homogeneous *triples* of geometry (0.1). Assuming that Komrakov's classification is correct, we focus on the common structure of such triples.

In Chapter 4, we define the homogeneous triples of type $\mathcal{H}_m^n$. We explicitly motivate the design, and derive the Riemannian curvature, and the Ricci tensor of the triples in $\mathcal{H}_m^n$. Notions such as Riemannian-, and Ricci-flatness reduce to non-linear equations in the parameters associated to a triple in $\mathcal{H}_m^n$.

According to a general result on semi-Riemannian manifolds, any homogeneous triple with geometry (0.1) is of dimension ≥ 4. In the last chapter, we discuss the properties of all 4-dimensional homogeneous triples that originate from our construction $\mathcal{H}_m^n$. This includes the case, where $\dim \mathfrak{h} = 0$. Subsequently, we confirm a small part of the classification of low-dimensional homogeneous triples in [DK95], which is originally stated without proof. Our efforts culminate in the (new) classification of all 4-dimensional homogeneous triples with $\dim \mathfrak{h} \geq 1$, and desired geometry.

Chapter 1

Propaedeutic

This chapter neither substitutes excellent literature on semi-Riemannian geometry such as [ON83], [Bo86], and [Bu85], nor appealing books on Lie theory and homogeneous spaces such as [HN91], and [Ar03]. Nevertheless, we briefly review the mathematical concepts that are relevant to our work.

Our notation does not deviate from what is commonly used in the literature. The formulas we state serve as a reference. Most of the examples have applications later on.

1.1 Lorentzian scalar products

Let V be a n-dimensional vector space. A *scalar product* is a symmetric non-degenerated bilinear form $(,) : V \times V \to \mathbb{R}$. Two vectors $v, w \in V$ are *orthogonal* if $(v, w) = 0$. A set of vectors $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ with $(\tilde{e}_i, \tilde{e}_j) \in \pm\delta_{i,j}$ forms an *orthonormal basis* of V. The *index* of a scalar product $(,)$ is the largest integer that is the dimension of a subspace $W \subset V$ on which $(,)$ is negative definite, i.e. $(w_1, w_2) \leq 0$ for all $w_1, w_2 \in W$.

To $(,) : V \times V \to \mathbb{R}$ we associate a matrix B relative to a basis of $V = \langle e_1, \dots, e_n \rangle$ so that $(u, v) = v^T.B.u$ for all $u, v \in V$. B is symmetric and $\det B \neq 0$, [ON83] p.47. The scalar products we are mostly dealing with in this thesis are of index 1, called *Lorentzian*[1]. In that case, we usually pick a basis of $V = \langle e_1, \dots, e_n \rangle$ so that the matrix associated to $(,)$ is of the form

$$B = L_n := \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{where} \quad I_n := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the identity matrix. The scalar product $(,)$ is negative definite on $W = \langle e_1 - e_n \rangle$, but not on any 2-dimensional subspace.

Example 1.1. Let $a, b, c, d \in \mathbb{R}$ and $a, b \neq 0$. The scalar product on $V = \langle e_1, \dots, e_4 \rangle$ defined by the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ a & 0 & c & d \end{pmatrix} \quad \text{with inverse} \quad B^{-1} = \frac{1}{a^2 b} \begin{pmatrix} c^2 - bd & 0 & -ac & ab \\ 0 & ab & 0 & 0 \\ -ac & 0 & a^2 & 0 \\ ab & 0 & 0 & 0 \end{pmatrix}$$

[1] Hendrik Antoon Lorentz, * 18. Jul 1853 in Arnhem, † 4. Feb 1928 in Haarlem

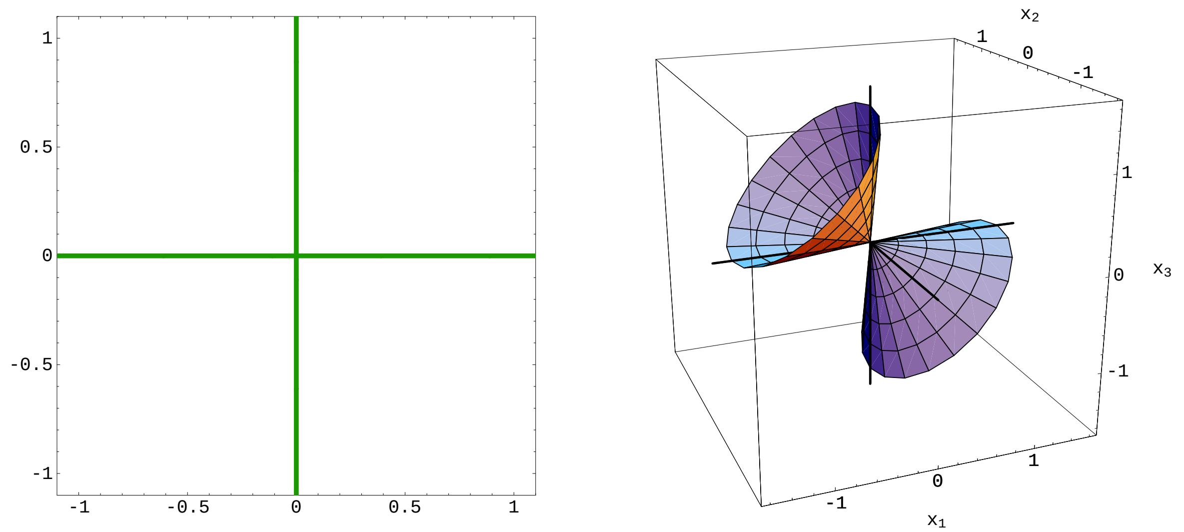

Figure 1.1 : For a Lorentzian scalar product with associated matrix B, the set $\{x \in V : x^T.B.x = 0\}$ defines a cone. We plot the sets $\{x \in \mathbb{R}^2 : x^T.L_2.x = 2x_1x_2 = 0\}$, and $\{x \in \mathbb{R}^3 : x^T.L_3.x = 2x_1x_3 + x_2^2 = 0\}$.

has index 1 for $a, b > 0$ positive, and index 3 for $a, b < 0$ negative. This becomes evident when we display B with respect to the basis $V = \langle \alpha_i^{-1}.e_1, \ldots, \alpha_i^{-1}.e_n \rangle$ for matrices α_i as

α_1	$\alpha_1^T.B.\alpha_1$	α_3	$\alpha_3^T.B.\alpha_3$
$\begin{pmatrix} \frac{1}{\sqrt{a}} & 0 & 0 & \frac{c^2-bd}{2a^{3/2}b} \\ 0 & \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{b}} & \frac{-c}{\sqrt{ab}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{a}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{-1}{\sqrt{-a}} & 0 & 0 & \frac{bd-c^2}{2(-a)^{3/2}b} \\ 0 & \frac{1}{\sqrt{-a}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{-b}} & \frac{-c}{\sqrt{-ab}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{-a}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

The determinant of B is $|B| = -a^3b$. The remaining combinations $b < 0 < a$, and $a < 0 < b$, lead to B of index 2. $\diamond$

Remark 1.2. Let $n \geq 1$. The equation $\alpha_i^T.L_{n+2}.\alpha_i = L_{n+2}$ holds for any

$$\alpha_1 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 1 & \eta^T & -\eta^T.\eta/2 \\ 0 & I_m & -\eta \\ 0 & 0 & 1 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 1 & 0 & 0 \\ -\eta & I_m & 0 \\ -\eta^T.\eta/2 & \eta^T & 1 \end{pmatrix}$$

where $\lambda \in \mathbb{R}^*$, $Q \in \mathbb{R}^{n\times n}$ with $Q^T.Q = Q.Q^T = I_n$, and $\eta \in \mathbb{R}^n$. In other words, a Lorentzian scalar product is *stable* under these types of vector space transformations. $\diamond$

Let B be a symmetric matrix with $\det B \neq 0$. Then B, and B^{-1} define two scalar products of the same index on V. A fact, easy to prove in matrix notation.

Lemma 1.3. Consider a vector space V of dimension $n \geq 2$ and a Lorentzian scalar product on V with associated matrix B. Let a skew symmetric matrix $\Omega \in \mathbb{R}^{n\times n}$ define a linear mapping $\Omega : V \to V$. The matrix product $-\Omega^T.B^{-1}.\Omega : V \to V$ maps all vectors to zero iff $\Omega = 0$.

Proof. In order to prove "$\Rightarrow$", we assume $\Omega \neq 0$. Since Ω is skew symmetric, the image of Ω contains a 2-dimensional subspace of V, i.e. $\dim \operatorname{im} \Omega \geq 2$. Because B corresponds to a Lorentzian scalar product, B^{-1} defines a scalar product of index 1 as well. Suppose $-\Omega^T.B^{-1}.\Omega = 0$, but then the set $\{x \in V : x^T.B^{-1}.x = 0\}$ contains a 2-dimensional vector space – a contradiction to B^{-1} being of index 1. □

1.2 Semi-Riemannian manifolds

In this thesis, *manifolds* M are differentiable manifolds of finite dimension. Functions and vector fields on manifolds are smooth. $\mathfrak{F}(M)$ denotes the set of real values functions $f : M \to \mathbb{R}$ on the manifold M. $\mathfrak{X}(M)$ denotes the set of vector fields on M. In this section, all notions and formulas are with respect to a single manifold M, so we abbreviate these sets to $\mathfrak{F}$, and $\mathfrak{X}$.

Remarkably, for two vector fields $U, V \in \mathfrak{X}$ there exists a unique third $W \in \mathfrak{X}$, which satisfies $Wf = U(Vf) - V(Uf)$ for all (smooth) functions $f \in \mathfrak{F}$, [Bo86] p.152. The *vector field commutator* is defined as $[U, V]_{\mathfrak{X}} := W$, or simply $[U, V]$.

A *linear connection* $D : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ with $D_V U := D(U, V)$ is a function that is $\mathbb{R}$-linear in the first slot, and *tensor-like* in the second, i.e.

$$D_V(U + W) = D_V U + D_V W, \quad \text{and} \quad D_{fV} U = f D_V U \quad \text{for all } U, V, W \in \mathfrak{X},\ f \in \mathfrak{F}.$$

Additionally, D satisfies the product rule

$$D_V(fU) = (Vf)U + f D_V U \quad \text{for all } U, V \in \mathfrak{X},\ f \in \mathfrak{F}.$$

The value of $D_V U \in \mathfrak{X}$ at each point $x \in M$ defines the vector rate of change of U in the V_x direction.

Please consult [ON83] pp.28 for the definitions of *integral curves*, and the *(local) flow* induced by a vector field $U \in \mathfrak{X}$. A vector field is *complete* if each of its maximal integral curves is defined on $\mathbb{R}$. Prominent examples of manifolds are Lie groups, and homogeneous spaces.

A *semi-Riemannian*[2] *manifold* (M, g) is a manifold M together with a symmetric non-degenerated $(0, 2)$-tensor field g on M. g is the *metric* on M, which evokes all geo*metric* notions such as isometry, geodesics, and curvature. Non-degeneracy of g means that, when restricted to a point $x \in M$, the metric defines a scalar product $g_x : T_xM \times T_xM \to \mathbb{R}$ on the *tangent (vector) space* T_xM of M in x. g_x has constant index for all $x \in M$, which we make the *index* of the metric g. A metric of index 0 is called *Riemannian*, a metric of index 1 is called *Lorentzian.* While on any manifold a Riemannian metric exists, we find the following statement in [ON83] p.149.

Theorem 1.4. *For a smooth manifold M the following are equivalent: (1) There exists a Lorentzian metric on M. (2) There is a non-vanishing vector field on M. (3) Either M is non-compact, or M is compact and has Euler number $\chi(M) = 0$.*

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[2] Georg Friedrich Bernhard Riemann, * 17. September 1826 in Breselenz, † 20. Juli 1866 in Selasca

A mapping $\alpha : M \to M$ is an *isometry* of M if the following diagram commutes

$$\begin{array}{llllll} g_x : & T_xM & \times & T_xM & \to & \mathbb{R} \\ & \downarrow \mathrm{d}\alpha|_x & & \downarrow \mathrm{d}\alpha|_x & & \uparrow \mathrm{Id} \qquad \text{for all } x \in M. \\ g_{\alpha x} : & T_{\alpha x}M & \times & T_{\alpha x}M & \to & \mathbb{R} \end{array}$$

The term $\mathrm{d}\alpha|_x$ abbreviates the differential $\mathrm{d}_y\alpha(y)|_{y=x}$, and $\alpha x = \alpha(x)$.

For two vector fields $U, V \in \mathfrak{X}$ it is convenient to write $\langle U, V\rangle := g(U, V)$. U and V are *orthogonal* if $\langle U, V\rangle = 0$. A vector field $W \in \mathfrak{X}$ is *Killing*[3] if the flow induced by W is an isometry for all stages of the flow, equivalently

$$W\langle U, V\rangle = \langle [W, U], V\rangle + \langle U, [W, V]\rangle \qquad \text{for all } U, V \in \mathfrak{X}. \tag{1.1}$$

On a semi-Riemannian manifold (M, g), there exists the unique *Levi-Civita*[4] linear connection $\nabla : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ with the additional properties

$$\nabla_U V - \nabla_V U = [U, V], \qquad \text{and} \qquad W\langle U, V\rangle = \langle D_W U, V\rangle + \langle U, \nabla_W V\rangle \qquad \text{for all } U, V, W \in \mathfrak{X}. \tag{1.2}$$

Combining the three relations in (1.1), (1.2), we yield for a Killing vector field $W \in \mathfrak{X}$

$$\langle \nabla_U W, V\rangle + \langle U, \nabla_V W\rangle = 0 \qquad \text{for all } U, V \in \mathfrak{X}. \tag{1.3}$$

The distinguished linear connection ∇ is characterized by the *Koszul*[5] *formula*

$$2\langle \nabla_V U, W\rangle = V\langle U, W\rangle + U\langle W, V\rangle - W\langle V, U\rangle - \langle V, [U, W]\rangle + \langle U, [W, V]\rangle + \langle W, [V, U]\rangle$$

for all $U, V, W \in \mathfrak{X}$. If these vector fields are moreover Killing, we make use of (1.1) to simplify the Koszul formula to

$$\begin{aligned} 2\langle \nabla_V U, W\rangle &= \langle [V, U], W\rangle + \langle U, [V, W]\rangle + \langle [U, W], V\rangle + \langle W, [U, V]\rangle - \langle [W, V], U\rangle \\ &\quad - \langle V, [W, U]\rangle - \langle V, [U, W]\rangle + \langle U, [W, V]\rangle + \langle W, [V, U]\rangle \\ &= -\langle [U, V], W\rangle - \langle U, [W, V]\rangle - \langle [W, U], V\rangle . \end{aligned} \tag{1.4}$$

The Theorema Egregium by F. Gauß[6] states that the Gauß-curvature of a surface depends solely on the first fundamental form, i.e. the metric. B. Riemann was inspired by this result and worked towards a generalization of curvature in higher dimensions.

The *Riemannian curvature* is the $(1, 3)$-tensor field $R : \mathfrak{X}^3 \to \mathfrak{X}$ defined by

$$R_{U,V}W := R(U, V, W) = \nabla_{[U,V]}W - \nabla_U\nabla_V W + \nabla_V\nabla_U W \qquad \text{for all } U, V, W \in \mathfrak{X}. \tag{1.5}$$

The following identities are immediate consequences of the Levi-Civita connection, and the Koszul formula. The intrinsic symmetries of R are

$$R_{U,V}W + R_{V,U}W = 0, \quad R_{U,V}W + R_{V,W}U + R_{W,U}V = 0, \quad \langle R_{U,V}W, X\rangle + \langle R_{U,V}X, W\rangle = 0, \tag{1.6}$$

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[3] Wilhelm Killing, * 10. May 1847 Burbach, † 11. Feb 1923 Münster

[4] Tullio Levi-Civita, * 29. Mar 1873 in Padua, † 29. Dec 1941 in Rom

[5] Jean-Louis Koszul, * 3. Jan 1921 in Strasbourg

[6] Johann Carl Friedrich Gauß, * 30. Apr 1777 in Braunschweig, † 23. Feb 1855 in Göttingen

for all $U, V, W, X \in \mathfrak{X}$, [Sa97] p.34. Combining two of the symmetries, we obtain

$$R_{U,V}W = R_{U,W}V - R_{V,W}U \quad \text{for all } U, V, W \in \mathfrak{X}. \tag{1.7}$$

According to [Be80] p.26, we have $R_{U,W}V = \nabla_{\nabla_W V}U - \nabla_W \nabla_V U$ for all $V, W \in \mathfrak{X}$ and any Killing vector field $U \in \mathfrak{X}$. Assume additionally V is Killing, then (1.7) becomes

$$\begin{aligned} R_{U,V}W &= \nabla_{\nabla_W V}U - \nabla_W \nabla_V U - \nabla_{\nabla_W U}V + \nabla_W \nabla_U V \\ &= \nabla_{\nabla_W V}U - \nabla_{\nabla_W U}V + \nabla_W [U, V]. \end{aligned} \tag{1.8}$$

The *Ricci*[7] *curvature* is the $(0, 2)$-tensor field $\mathrm{Ric} : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{F}$ defined by

$$\mathrm{Ric}(U, V) = \operatorname{tr}(W \mapsto R_{W,U}V) \quad \text{for all } U, V \in \mathfrak{X}. \tag{1.9}$$

According to [Sa97] p.44, Ric is symmetric, i.e. $\mathrm{Ric}(U, V) = \mathrm{Ric}(V, U)$ for all $U, V \in \mathfrak{X}$.

A semi-Riemannian manifold is *Ricci-flat* if $\mathrm{Ric} = 0$. An *Einstein*[8] manifold carries a metric g such that $\mathrm{Ric} = \lambda g$ for some constant $\lambda \in \mathbb{R}$.

1.3 Lie groups and Lie algebras

Let G be a manifold and a group. G is a *Lie*[9] *group*, if the group operation $\circ : G \times G \to G$ and the inverse mapping $\zeta : G \to G$ are smooth. We denote the neutral element as $\mathrm{e} \in G$.

An omnipresent Lie group is the *general linear group* $\mathrm{GL}(\mathbb{R}^n)$, which is the set (and differentiable manifold) of all automorphisms of the vector space $\mathbb{R}^n$. $\mathrm{GL}(\mathbb{R}^n)$ is identified with the set of all invertible $(n \times n)$-matrices with coefficients in $\mathbb{R}$. The group operation is matrix multiplication. The inverse ζ is matrix inversion. The neutral element e is the identity matrix $I_n \in \mathrm{GL}(\mathbb{R}^n)$. The dimension of $\mathrm{GL}(\mathbb{R}^n)$ is n^2.

The isometries of a semi-Riemannian manifold (M, g) form the Lie group $\mathrm{Aut}(M, g)$. The group operation is concatenation of isometries. The inverse of an isometry is the inverse diffeomorphism. The neutral element is the identity mapping on the manifold. If M is complete, we have

$$\dim \mathrm{Aut}(M, g) \leq \frac{m(m+1)}{2}, \tag{1.10}$$

where $m = \dim M$, [Sa97] p.120.

An immersed submanifold H of G that is also an abstract subgroup of G is called a *Lie subgroup*.

Example 1.5. The set D_+ of all $(n \times n)$-matrices with strictly positive entries on the diagonal is a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$. D_+ is isomorphic to the Lie group $(\mathbb{R}^n, +)$, the space $\mathbb{R}^n$ with vector addition. An isomorphism is given by $(x_1, \dots, x_n) \mapsto \mathrm{diag}(\exp x_1, \dots, \exp x_n)$.

The set of all $(n \times n)$-matrices with determinant 1 forms the *special linear group* $\mathrm{SL}(\mathbb{R}^n)$, which again is a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$. ◇

[7] Gregorio Ricci-Curbastro, * 12. Jan 1853 in Lugo/Ravenna, † 6. Aug 1925 in Bologna

[8] Albert Einstein, * 14. Mar 1879 in Ulm, † 18. Apr 1955 in Princeton

[9] Marius Sophus Lie, * 17. Dec 1842 in Nordfjordeid, † 18. Feb 1899 in Oslo

The *left-translation* by $x \in G$ is the diffeomorphism $L_x : G \to G$, which maps $y \mapsto xy$. A vector field $U \in \mathfrak{X}(G)$ on G is *left-invariant* if $\mathrm{d}L_{\zeta(x)}|_x U_x = U_e$ for all $x \in G$. Such a vector field is determined by $u \in T_{\mathrm{e}}G$ thru $\bar{u}_x = \mathrm{d}L_x|_{\mathrm{e}} u$. The term $\mathrm{d}L_x|_{\mathrm{e}}$ abbreviates the differential $\mathrm{d}_y L_x(y)|_{y=\mathrm{e}}$. In general, to each (r,s)-tensor A on $T_{\mathrm{e}}G$, there corresponds a unique *left-invariant* (r,s)-tensor field $\bar{A}$ on G with $\bar{A}_e = A$. The scheme for a $(1,2)$-tensor is

$$\begin{array}{rcccccl} \bar{A}_x: & T_xG & \times & T_xG & \to & T_xG & \\ & \downarrow \mathrm{d}L_{\zeta(x)}|_x & & \downarrow \mathrm{d}L_{\zeta(x)}|_x & & \uparrow \mathrm{d}L_x|_{\mathrm{e}} & \text{for all } x \in G. \\ A: & T_{\mathrm{e}}G & \times & T_{\mathrm{e}}G & \to & T_{\mathrm{e}}G & \end{array}$$

A brief computation reveals, that the commutator $[\bar{u}, \bar{v}]_{\mathfrak{X}}$ of two left-invariant vector fields $\bar{u}, \bar{v} \in \mathfrak{X}(G)$ again is a left-invariant vector field. This closeness leads to the following notion: The *Lie algebra* $\mathfrak{g}$ of a Lie group G is the vector space $T_{\mathrm{e}}G$ together with a $(1,2)$-tensor $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called *Lie bracket*, or *commutator*. The Lie bracket is defined as $[u,v] := [\bar{u}, \bar{v}]_{\mathfrak{X}}|_{\mathrm{e}}$ for all vectors $u, v \in \mathfrak{g}$. The commutator is skew symmetric, i.e. $[u,v] = -[v,u]$, and satisfies the *Jacobi*[10] *identity*,

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0 \quad \text{for all } u,v,w \in \mathfrak{g}. \tag{1.11}$$

A left-invariant vector field $\bar{u} \in \mathfrak{X}(G)$ induces a unique integral curve $\gamma_u : \mathbb{R} \to G$ such that $\gamma_u(0) = \mathrm{e}$, [Ar03] p.16. The *exponential map* is $\exp : \mathfrak{g} \to G$ with $u \mapsto \gamma_u(1)$.

The Lie algebra of $\mathrm{GL}(\mathbb{R}^n)$ is $\mathfrak{gl}(\mathbb{R}^n)$, the set of all $(n \times n)$-matrices. Two matrices $X, Y \in \mathfrak{gl}(\mathbb{R}^n)$ commute as $[X,Y]_{\mathfrak{gl}} = X.Y - Y.X$, [HN91] p.26. The exponential map corresponds to the *matrix exponential* $\exp_{\mathfrak{gl}} : \mathfrak{gl}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$ with $\exp_{\mathfrak{gl}}(X) = \sum_{k \in \mathbb{N}_0} X^k/k!$.

Let (M,g) be a semi-Riemannian manifold. The Lie algebra of $\mathrm{Aut}(M,g)$ is the set of all complete Killing vector fields on M, [ON83] p.255.

A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. Then, $\mathfrak{h}$ is a Lie algebra with commutator induced by $\mathfrak{g}$ restricted to elements of $\mathfrak{h}$.

Example 1.6. The set $\mathfrak{d}$ of all $(n \times n)$-matrices with entries on the diagonal is the Lie algebra of the Lie group D_+ and a Lie subalgebra of $\mathfrak{gl}(\mathbb{R}^n)$. Any two diagonal matrices $X, Y \in \mathfrak{d}$ *commute*, i.e. the commutator $[X,Y]_{\mathfrak{gl}} = X.Y - Y.X = 0$ vanishes.

The Lie algebra of $\mathrm{SL}(\mathbb{R}^n)$ is denoted $\mathfrak{sl}(\mathbb{R}^n)$. The algebra consists of all $(n \times n)$-matrices with trace 0. $\mathfrak{sl}(\mathbb{R}^n)$ again is a Lie subalgebra of $\mathfrak{gl}(\mathbb{R}^n)$. For instance, $\mathfrak{sl}(\mathbb{R}^2) = \langle X_1, X_2, X_3 \rangle$ is the 3-dimensional vector space spanned by the matrices

$$X_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{with} \quad [,]_{\mathfrak{gl}} = \begin{array}{c|ccc} & X_1 & X_2 & X_3 \\ \hline X_1 & 0 & X_2 & -X_3 \\ X_2 & -X_2 & 0 & 2X_1 \\ X_3 & X_3 & -2X_1 & 0 \end{array}.$$

By linearity, the commutator is determined by the values on elements of the basis. ◇

[10] Carl Gustav Jacob Jacobi, * 10. Dec 1804 in Potsdam, † 18. Feb 1851 in Berlin

Beispiele. Eine bekannte Gruppe dieser Art ist die folgende:

$$x' = \frac{x + a_1}{a_2 x + a_3},$$

welche drei Parameter a_1, a_2, a_3 enthält. Führt man die beiden Transformationen

$$x' = \frac{x + a_1}{a_2 x + a_3}, \quad x'' = \frac{x' + b_1}{b_2 x' + b_3}$$

nach einander aus, so erhält man:

$$x'' = \frac{x + c_1}{c_2 x + c_3},$$

wo c_1, c_2, c_3 als Functionen von den a und b durch die Relationen

$$c_1 = \frac{a_1 + b_1 a_3}{1 + b_1 a_2}, \quad c_2 = \frac{b_2 + a_2 b_3}{1 + b_1 a_2}, \quad c_3 = \frac{b_2 a_1 + b_3 a_3}{1 + b_1 a_2}$$

definirt sind.

Figure 1.2 : Excerpt from *Theorie der Transformationsgruppen*, [Lie30] p.4. The author yields the group action of $\mathrm{SL}(\mathbb{R}^2)$ as $(a_1, a_2, a_3) \circ (b_1, b_2, b_3) = (c_1, c_2, c_3)$ on a neighborhood of $\mathrm{e} = (0, 0, 1)$.

Two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ are *isomorphic* if there exists a vector space isomorphism $\alpha : \mathfrak{g}_2 \to \mathfrak{g}_1$ satisfying $[u, v]_{\mathfrak{g}_2} = \alpha^{-1}.[\alpha.u, \alpha.v]_{\mathfrak{g}_1}$ for all $u, v \in \mathfrak{g}_2$. The corresponding diagram is

$$\begin{array}{ccccccc} [,]_{\mathfrak{g}_2} : & \mathfrak{g}_2 & \times & \mathfrak{g}_2 & \to & \mathfrak{g}_2 & \\ & \downarrow \alpha & & \downarrow \alpha & & \uparrow \alpha^{-1} & . \\ [,]_{\mathfrak{g}_1} : & \mathfrak{g}_1 & \times & \mathfrak{g}_1 & \to & \mathfrak{g}_1 & \end{array}$$

The Lie group $\mathrm{Aut}(\mathfrak{g})$ is the group of all automorphisms $\alpha \in \mathrm{GL}(\mathfrak{g})$ that leave the commutator of $\mathfrak{g}$ invariant as $[u, v]_{\mathfrak{g}} = \alpha^{-1}.[\alpha.u, \alpha.v]_{\mathfrak{g}}$ for all $u, v \in \mathfrak{g}$. The Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is the set of *derivations*

$$\mathrm{Der}(\mathfrak{g}) := \{\delta \in \mathfrak{gl}(\mathfrak{g}) : \delta.[u, v] - [\delta.u, v] - [u, \delta.v] = 0 \text{ for all } u, v \in \mathfrak{g}\}. \tag{1.12}$$

Conjugation by $x \in G$ is the mapping $\Xi_x : G \to G$ with $y \mapsto xyx^{-1}$. The *adjoint representation* of G is the group homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ given by $x \mapsto \mathrm{d}\Xi_x|_{\mathrm{e}}$. The *adjoint representation* of $\mathfrak{g}$ is the homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ given by $\mathrm{ad} = \mathrm{dAd}|_{\mathrm{e}}$. The $(1, 2)$-tensor $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is identical to the Lie bracket $[,]$.

The *descending series* of a Lie algebra $\mathfrak{g}$ is the sequence of subspaces $\mathfrak{g}^i \subset \mathfrak{g}$ defined iteratively as

$$[\mathfrak{g}]^1 := \mathfrak{g}, \qquad \text{and} \qquad [\mathfrak{g}]^{i+1} := [\mathfrak{g}, [\mathfrak{g}]^i] \qquad \text{for } i \in \mathbb{N}.$$

A Lie algebra is *k-step nilpotent* if $[\mathfrak{g}]^k \neq [\mathfrak{g}]^{k+1} = \{0\}$. A k-step nilpotent algebra $\mathfrak{n}$ is the Lie algebra of the group $(\mathfrak{n}, \circ_{\mathrm{cbh}})$, where the group action $\circ_{\mathrm{cbh}}$ on $\mathfrak{n}$ is given by the Campbell-Baker-Hausdorff series

$$x \circ_{\mathrm{cbh}} y = x + y + \frac{1}{2}[x, y] - \frac{1}{12}\big([[x, y], x] + [[y, x], y]\big) + \frac{1}{24}[[[y, x], x], y] + \ldots \tag{1.13}$$

The inverse $\zeta : \mathfrak{n} \to \mathfrak{n}$ maps $x \mapsto -x$, and the neutral element has the coordinates $\mathrm{e} = (0, \ldots, 0)$. Vector addition other than (1.13) is not a meaningful operation on the Lie group $(\mathfrak{n}, \circ_{\mathrm{cbh}})$.

Expression (1.13) lists the first terms of an infinite series, [HN91] p.44. However, for a k-step nilpotent Lie algebra summands involving the commutator of order $\geq k$ vanish. For many important Lie groups such as the *orthogonal group* $\mathrm{O}_n = \{x \in \mathrm{GL}(\mathbb{R}^n) : x^T.x = I_n\}$ for instances with $n \geq 3$, an explicit formula of the group operation does not exist in local coordinates. In the neighborhood of e, the first terms of the Campbell-Baker-Hausdorff series usually provide a good numerical approximation of the group action.

Example 1.7. Let $n \in \mathbb{N}_0$. The *n-Heisenberg*[11] *group* He_n is a $2n+1$-dimensional Lie subgroup of $\mathrm{GL}(\mathbb{R}^{n+2})$ consisting of the matrices

$$\mathrm{He}_n := \left\{ \begin{pmatrix} 1 & p^T & h \\ 0 & I_n & q \\ 0 & 0 & 1 \end{pmatrix} : p, q \in \mathbb{R}^n, h \in \mathbb{R} \right\} \subset \mathrm{GL}(\mathbb{R}^{n+2}). \tag{1.14}$$

The group operation of He_n is matrix multiplication, the inverse mapping is matrix inversion. As a manifold He_n is diffeomorphic to $\mathbb{R}^{2n+1} = \langle p_1, \ldots, p_n, h, q_1, \ldots, q_n \rangle$. Encoding the matrices (1.14) as triples $(p, h, q) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, the group action is

$$(x_p, x_h, x_q) \circ (y_p, y_h, y_q) = (x_p + y_p, x_h + y_h + x_p^T.y_q, x_q + y_q).$$

Inversion ζ maps $(x_p, x_h, x_q) \mapsto (-x_p, -x_h + x_p^T.x_q, -x_q)$. The triple $\mathrm{e} = (0, 0, 0)$ corresponds to the identity matrix.

The Lie algebra to He_n is the *n-Heisenberg algebra* $\mathfrak{he}_n$. Naturally, $\mathfrak{he}_n$ is a Lie subalgebra of $\mathfrak{gl}(\mathbb{R}^{n+2})$ consisting of the matrices

$$\mathfrak{he}_n := \left\{ \begin{pmatrix} 0 & p^T & h \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix} : p, q \in \mathbb{R}^n, h \in \mathbb{R} \right\} \subset \mathfrak{gl}(\mathbb{R}^{n+2}). \tag{1.15}$$

The commutator of $\mathfrak{he}_n$ is the matrix commutator. However, encoding the elements of the algebra $\mathfrak{he}_n$ as triples $(p, h, q) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ reduces the commutator to

$$[(u_p, u_h, u_q), (v_p, v_h, v_q)] = (0, u_p^T.v_q - v_p^T.u_q, 0). \tag{1.16}$$

We denote the canonic basis of $\mathfrak{he}_n$ as $\langle p_1, \ldots, p_n, h, q_1, \ldots, q_n \rangle$. Using (1.16), the only non-zero commutators of basis elements are $[p_i, q_i] = -[q_i, p_i] = h$ for $i = 1, \ldots, n$. We summarize the commutator relations of $\mathfrak{he}_1$, and $\mathfrak{he}_2$ as follows:

$$[,]_{\mathfrak{he}_1} = \begin{array}{c|ccc} & p_1 & h & q_1 \\ \hline p_1 & 0 & 0 & h \\ h & 0 & 0 & 0 \\ q_1 & -h & 0 & 0 \end{array} \qquad [,]_{\mathfrak{he}_2} = \begin{array}{c|ccccc} & p_1 & p_2 & h & q_1 & q_2 \\ \hline p_1 & 0 & 0 & 0 & h & 0 \\ p_2 & 0 & 0 & 0 & 0 & h \\ h & 0 & 0 & 0 & 0 & 0 \\ q_1 & -h & 0 & 0 & 0 & 0 \\ q_2 & 0 & -h & 0 & 0 & 0 \end{array}$$

[11] Werner Karl Heisenberg, * 5. Dec 1901 in Würzburg, † 1. Feb 1976 in München

For $n \geq 1$, we have $[\mathfrak{hx}_n]^2 = h\mathbb{R}$, and $[\mathfrak{hx}_n]^3 = \{0\}$. Thus, the n-Heisenberg algebra $\mathfrak{hx}_n$ is 2-step nilpotent. The Campbell-Baker-Hausdorff series reduces to $x \circ_{\text{cbh}} y = x + y + \frac{1}{2}[x, y]$. In triple notation, we obtain

$$(x_p, x_h, x_q) \circ_{\text{cbh}} (y_p, y_h, y_q) = \big(x_p + y_p, x_h + y_h + \big(x_p^T.y_q - y_p^T.x_q\big)/2, x_q + y_q\big)\,.$$

$\diamond$

Consider two Lie groups K, G, and a Lie group homomorphism $\Psi : K \to \text{Aut}(G)$. The *semi-direct product* $G \rtimes_\Psi K$ is the manifold $G \times K$ with group action

$$(x_1, y_1) \circ (x_2, y_2) = \big(x_1 \circ_G \Psi(y_1)(x_2), y_1 \circ_K y_2\big) \quad \text{ for } x_i \in G,\ y_i \in K.$$

We denote by $\mathfrak{g}, \mathfrak{k}$ the Lie algebras of G, K. The Lie algebra of $G \rtimes_\Psi K$ is the vector space $\mathfrak{g} \oplus \mathfrak{k}$ with commutator

$$[u_1 + v_1, u_2 + v_2] = [u_1, u_2]_\mathfrak{g} + [v_1, v_2]_\mathfrak{k} + (\mathrm{d}\psi|_\mathrm{e}.v_1).u_2 - (\mathrm{d}\psi|_\mathrm{e}.v_2).u_1 \quad \text{ for } u_i \in \mathfrak{g},\ v_i \in \mathfrak{k}, \tag{1.17}$$

where $\psi : K \to \text{Aut}(\mathfrak{g})$ with $y \mapsto \mathrm{d}_x\Psi(y)(x)|_{x=\mathrm{e}\in G}$. Then, $\mathrm{d}\psi|_{\mathrm{e}\in K}.v$ is a derivation on $\mathfrak{g}$ for all $v \in \mathfrak{k}$, [HN91] p.223.

Conversely, for a homomorphism $\Delta : \mathfrak{k} \to \text{Der}(\mathfrak{g})$ we define the *semi-direct product* Lie algebra with commutator as in (1.17) with $\mathrm{d}\psi|_\mathrm{e}$ replaced by Δ.

The Lie algebras most relevant for this thesis are semi-direct products of a nilpotent Lie algebra $\mathfrak{n}$ and a 1-dimensional algebra, which we denote by $\mathbb{R}$. Formally, we have $\mathfrak{n} \rtimes_\Delta \mathbb{R}$ with $\Delta : \mathbb{R} \to \text{Der}(\mathfrak{n})$. However, the mapping Δ is determined by a single matrix, say $\delta := \Delta(1) \in \text{Der}(\mathfrak{n})$. We denote by $\mathfrak{n} \rtimes_\delta \mathbb{R}$ the Lie algebra with commutator

$$[u_1 + v_1, u_2 + v_2] = [u_1, u_2]_\mathfrak{n} + v_1\, \delta.u_2 - v_2\, \delta.u_1 \quad \text{ for } u_i \in \mathfrak{n},\ v_i \in \mathbb{R}. \tag{1.18}$$

The commutator (1.18) satisfies the Jacobi identity iff $\delta : \mathfrak{n} \to \mathfrak{n}$ is a derivation on the Lie algebra $\mathfrak{n}$.

At the beginning of this section, we have indicated how to extend a tensor A of arbitrary tensor rank on $\mathfrak{g} = T_\mathrm{e}G$ to the corresponding left-invariant tensor field $\bar{A}$ on G. The left-translation $L_x : G \to G$ with $y \mapsto xy$ is a diffeomorphism for all $x \in G$. Accordingly, the mapping $\mathrm{d}L_x|_\mathrm{e} : T_\mathrm{e}G \to T_xG$ is a vector space isomorphism for all $x \in G$. Thus, the smooth $(0, 2)$-tensor field $\bar{B}$ originating from a scalar product B on $T_\mathrm{e}G$ defines a scalar product at every point $x \in G$, i.e. $\bar{B}$ is a metric on G.

$$\begin{array}{rlclcll} \bar{B}_x : & T_xG & \times & T_xG & \to & \mathbb{R} & \\ & \downarrow \mathrm{d}L_{\zeta(x)}|_x & & \downarrow \mathrm{d}L_{\zeta(x)}|_x & & \uparrow \mathrm{Id} & \text{for all } x \in G. \\ B : & T_\mathrm{e}G & \times & T_\mathrm{e}G & \to & \mathbb{R} & \end{array}$$

The tuple $(G, \bar{B})$ defines a semi-Riemannian manifold. The left-translation $L_x : G \to G$ is an isometry for all $x \in G$. Just as the metric $\bar{B}$ is left-invariant, all geometric tensor fields are left-invariant. For instance, the Riemannian curvature is $\bar{R}_\mathrm{o}$ for an appropriate $(1, 3)$-tensor R_o on $T_\mathrm{e}G$. In the next section, we yield R_o in terms of the commutator tensor ad, and the scalar product B on $\mathfrak{g}$ in the more general context of homogeneous triples.

1.4 Homogeneous spaces, pairs and triples

A subgroup $H \subset G$ of a Lie group G that is also a closed subset of G, is a *closed Lie subgroup.*

A *homogeneous space* is the coset manifold $G/H = \{xH : x \in G\}$, where G is a Lie group, and $H \subset G$ is a closed Lie subgroup, together with the mappings

$$\begin{array}{llll} \textit{projection} \text{ to } G/H & \pi : G \to G/H & \text{with} & x \mapsto xH, \\ \textit{left-action} \text{ on } G/H & \tau : G \times G/H \to G/H & \text{with} & (x, yH) \mapsto xyH. \end{array}$$

The following theorem from [Bo86] p.166 allows us to treat the factor space G/H as a differentiable manifold, and π, τ as smooth mappings.

Theorem 1.8. *There exists a unique C^∞-manifold structure on the space G/H with the properties: (i) π is smooth and (ii) $x \in G$ is in the image $\sigma(V)$ of a C^∞ section V, σ on G/H.*

The natural action $\tau : G \times G/H \to G/H$ is a smooth action of G on G/H with this structure. The dimension of G/H is $\dim G - \dim H$.

Analogous to $L_x : G \to G$ with $y \mapsto xy$ for $x \in G$, we specialize the left-action to $\tau_x : G/H \to G/H$ with $yH \mapsto xyH$ for $x \in G$. The identities $\pi(xy) = \tau_x(\pi y)$, and $\tau_x \tau_y(zH) = \tau_{xy}(zH)$ for all $x, y, z \in G$ are immediate consequences of the definition. We call H the *isotropy group.*

For $u \in \mathfrak{g}$ we define the *fundamental vector field* $\tilde{u} \in \mathfrak{X}(G/H)$ by

$$\tilde{u}_{xH} = \mathrm{d}_t \tau(\exp tu, xH)|_{t=0} \qquad \text{for all } xH \in G/H, \tag{1.19}$$

where $\exp : \mathfrak{g} \to G$ is the exponential mapping. $\tilde{u}$ is a Killing vector field on G/H, and $[\tilde{u}, \tilde{v}]_{\mathfrak{X}} = -\widetilde{[u, v]}_{\mathfrak{g}}$ for all $u, v \in \mathfrak{g}$, [ON83] p.255.

On a Lie group G, we made use of the vector space isomorphisms $\mathrm{d}L_{\zeta(x)}|_x$, and $\mathrm{d}L_x|_\mathrm{e}$ for all $x \in G$ to extend an (r, s)-tensor A on $\mathfrak{g} = T_\mathrm{e}G$ to the left-invariant tensor field $\bar{A}$ on G. There are no restrictions on the tensor A. The framework of homogeneous spaces provides the vector space isomorphisms

$$\mathrm{d}\tau_{\zeta(x)}|_{xH} : T_{xH}G/H \to T_\mathrm{o}G/H, \quad \text{and} \quad \mathrm{d}\tau_x|_\mathrm{o} : T_\mathrm{o}G/H \to T_{xH}G/H \quad \text{for all } x \in G,$$

where $\mathrm{o} = \mathrm{e}H$ denotes the distinguished coset. The scheme for a (1,2)-tensor A on $T_\mathrm{o}G/H$ is

$$\begin{array}{rccccc} \bar{A}_{xH} : & T_{xH}G/H & \times & T_{xH}G/H & \to & T_{xH}G/H \\ & \downarrow \mathrm{d}\tau_{\zeta(x)}|_{xH} & & \downarrow \mathrm{d}\tau_{\zeta(x)}|_{xH} & & \uparrow \mathrm{d}\tau_x|_\mathrm{o} \\ A : & T_\mathrm{o}G/H & \times & T_\mathrm{o}G/H & \to & T_\mathrm{o}G/H \end{array} \qquad \text{for all } x \in G.$$

However, if $H \neq \{\mathrm{e}\}$ the choice of $y \in G$ with $\tau_{\zeta(y)}(xH) = \mathrm{o}$ is not unique. In fact, any group element $y = xh$ with $h \in H$ maps $\tau_{\zeta(y)}(xH) = \zeta(h)\zeta(x)xH = \mathrm{o}$. The diagram is well defined iff

$$\begin{array}{rccccc} A : & T_\mathrm{o}G/H & \times & T_\mathrm{o}G/H & \to & T_\mathrm{o}G/H \\ & \downarrow \mathrm{d}\tau_{\zeta(h)}|_\mathrm{o} & & \downarrow \mathrm{d}\tau_{\zeta(h)}|_\mathrm{o} & & \uparrow \mathrm{d}\tau_h|_\mathrm{o} \\ A : & T_\mathrm{o}G/H & \times & T_\mathrm{o}G/H & \to & T_\mathrm{o}G/H \end{array} \qquad \text{for all } h \in H$$

commutes. Any (r,s)-tensor A on $T_{\mathrm{o}}G/H$ that is stable under the change of basis by $\mathrm{d}\tau_h|_{\mathrm{o}}$ for all $h \in H$ extends to the unique *left-invariant* tensor field $\bar{A}$ on G/H by the procedure illustrated above.

Left-invariant metrics $\bar{B}$ on a homogeneous space G/H are of particular interest. Such a metric originates from a single scalar product B on $T_{\mathrm{o}}G/H$. The tuple $(G/H, \bar{B})$ defines a semi-Riemannian manifold. The Koszul formula (1.4) restricted to fundamental vector fields becomes

$$\begin{aligned} 2\,\langle \nabla_{\tilde{v}}\tilde{u}, \tilde{w}\rangle &= -\,\langle [\tilde{u},\tilde{v}]_{\mathfrak{X}}, \tilde{w}\rangle - \langle \tilde{u}, [\tilde{w},\tilde{v}]_{\mathfrak{X}}\rangle - \langle [\tilde{w},\tilde{u}]_{\mathfrak{X}}, \tilde{v}\rangle \\ &= \langle \widetilde{[u,v]_{\mathfrak{g}}}, \tilde{w}\rangle + \langle \tilde{u}, \widetilde{[w,v]_{\mathfrak{g}}}\rangle + \langle \widetilde{[w,u]_{\mathfrak{g}}}, \tilde{v}\rangle \quad \text{for all } u,v,w \in \mathfrak{g}. \end{aligned} \tag{1.20}$$

According to (1.8), the Riemannian curvature simplifies to

$$\begin{aligned} R_{\tilde{u},\tilde{v}}\tilde{w} &= \nabla_{\nabla_{\tilde{w}}\tilde{v}}\tilde{u} - \nabla_{\nabla_{\tilde{w}}\tilde{u}}\tilde{v} + \nabla_{\tilde{w}}[\tilde{u},\tilde{v}]_{\mathfrak{X}} \\ &= \nabla_{\nabla_{\tilde{w}}\tilde{v}}\tilde{u} - \nabla_{\nabla_{\tilde{w}}\tilde{u}}\tilde{v} - \nabla_{\tilde{w}}\widetilde{[u,v]_{\mathfrak{g}}} \quad \text{for all } u,v,w \in \mathfrak{g}. \end{aligned} \tag{1.21}$$

The curvature is left-invariant, i.e. $R = \bar{R}_{\mathrm{o}}$. Properties such as Riemannian-, and Ricci-flatness deduce from R_{o}.

The *homogeneous pair* associated to a homogeneous space G/H is the tuple $(\mathfrak{g},\mathfrak{h})$, where $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{h} \subset \mathfrak{g}$ is the maximal Lie subalgebra with $\exp \mathfrak{h} \subset H$. By $\mathfrak{m}$, we denote a vector space complement so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Since H is a closed Lie subgroup, the homogeneous space G/H can be locally parametrized by a submanifold M of G that consists of the points $\exp m \in G$ for vectors $m \in \mathfrak{m}$ in a neighborhood of $0 \in \mathfrak{m}$. We identify $\mathfrak{m} = T_{\mathrm{e}}M = T_{\mathrm{o}}G/H$, and $M \simeq M/H \subset G/H$ as an open subset of the homogeneous space.

We agree on the following convention. A linear mapping $A : \mathfrak{g} \to \mathfrak{g}$ induces $A|_V : V \to \mathfrak{g}$, and $A_V : V \to V$ for a subspace $V \subset \mathfrak{g}$. $A|_V$ is just the restriction to elements in V, whereas $A_V = \pi_V \circ A|_V$.

The transformation of $T_{\mathrm{o}}G/H$ by $\mathrm{d}\tau_h|_{\mathrm{o}}$ for $h \in H$ corresponds to the transformation of $\mathfrak{m}$ by $\mathrm{Ad}(h)_{\mathfrak{m}}$ for $h \in H$. The following diagram commutes

$$\begin{array}{llll} T_{\mathrm{o}}G/H \;\rightarrow & \mathfrak{g}/\mathfrak{h} & \rightarrow\; \mathfrak{m} & \\ \uparrow \mathrm{d}\tau_h|_{\mathrm{o}} & \uparrow \hat{\mathrm{Ad}}(h) & \uparrow \mathrm{Ad}(h)_{\mathfrak{m}} & \text{for all } h \in H, \\ T_{\mathrm{o}}G/H \;\rightarrow & \mathfrak{g}/\mathfrak{h} & \rightarrow\; \mathfrak{m} & \end{array}$$

where $\hat{\mathrm{Ad}} : H \to \mathrm{Aut}(\mathfrak{g}/\mathfrak{h})$ is the induced adjoint representation on the quotient $\mathfrak{g}/\mathfrak{h}$, since $\mathfrak{h}$ is $\mathrm{Ad}(H)$-invariant. This motivates the *linear isotropy group* I, and the *linear isotropy algebra* $\mathfrak{i}$, as

$$I = \{\mathrm{Ad}(h)_{\mathfrak{m}} : h \in H\} \subset \mathrm{GL}(\mathfrak{m})$$

$$\mathfrak{i} = \{\mathrm{ad}(h)_{\mathfrak{m}} : h \in \mathfrak{h}\} = \{m \mapsto [h,m]_{\mathfrak{m}} : h \in \mathfrak{h}\} \subset \mathfrak{gl}(\mathfrak{m}).$$

We call tensors on $\mathfrak{m}$ *ρ-invariant* that are $\mathrm{Ad}(h)_{\mathfrak{m}}$-invariant for all $h \in H$. For instance, denote with B, C tensors on $\mathfrak{m}$ of rank (0,2), and (1,2). Then, B, C are ρ-invariant if

$$B(\alpha.u, \alpha.v) = B(u,v), \quad \text{and} \quad \alpha^{-1}.C(\alpha.u, \alpha.v) = C(u,v) \quad \text{for all } u,v \in \mathfrak{m},\ \alpha \in I,$$

Equivalently, B, C on $\mathfrak{m}$ are ρ-invariant if

$$B(X.u, v) + B(u, X.v) = 0,$$
$$-X.C(u, v) + C(X.u, v) + C(u, X.v) = 0 \quad \text{for all } u, v \in \mathfrak{m},\ X \in \mathfrak{i}.$$

The *isotropy representation* is $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m})$ with $\rho(h) = \mathrm{ad}(h)_{\mathfrak{m}}$ for $h \in \mathfrak{h}$.

A ρ-invariant tensor on $\mathfrak{m}$ uniquely extends to a left-invariant tensor field on G/H. Denote with $\bar{B}$ the left-invariant metric on G/H, that originates from a ρ-invariant scalar product B on $\mathfrak{m}$. We consider the semi-Riemannian manifold $(G/H, \bar{B})$. In the sequel, we derive a formula for the Riemannian curvature tensor R_{o} at $\mathrm{o} \in G/H$ in terms of the commutator of $\mathfrak{g}$, and the scalar product B on $\mathfrak{m}$.

We introduce the *Levi-Civita connection tensor* $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ with

$$\Lambda(u).v_{\mathfrak{m}} = (\nabla_{\tilde{v}} \tilde{u})_{\mathrm{o}} \quad \text{for all } u, v \in \mathfrak{g},$$

which represents the Levi-Civita connection ∇ in the point $\mathrm{o} \in G/H$. According to (1.20),

$$2B(\Lambda(u).v_{\mathfrak{m}}, w_{\mathfrak{m}}) = B([u, v]_{\mathfrak{m}}, w_{\mathfrak{m}}) + B(u_{\mathfrak{m}}, [w, v]_{\mathfrak{m}}) + B([w, u]_{\mathfrak{m}}, v_{\mathfrak{m}}) \quad \text{for all } u, v, w \in \mathfrak{g}.$$

The tensor Λ apparently decomposes as

$$\Lambda(u).v_{\mathfrak{m}} = \frac{1}{2}[u, v]_{\mathfrak{m}} + \nu(u, v) \quad \text{for all } u, v \in \mathfrak{g}, \tag{1.22}$$

where $\nu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ is uniquely determined by

$$2B\big(\nu(u, v), w_{\mathfrak{m}}\big) = B\big(u_{\mathfrak{m}}, [w, v]_{\mathfrak{m}}\big) + B\big([w, u]_{\mathfrak{m}}, v_{\mathfrak{m}}\big) \quad \text{for all } u, v, w \in \mathfrak{g}. \tag{1.23}$$

Interchanging u, v in the rhs. and using that B is symmetric gives the symmetry $\nu(u, v) = \nu(v, u)$ for all $u, v \in \mathfrak{g}$. Let $h, h_1, h_2 \in \mathfrak{h}$, $m \in \mathfrak{m}$, and $w \in \mathfrak{g}$. We yield

$$2B\big(\nu(h_1, h_2), w_{\mathfrak{m}}\big) = B\big(0, [w, h_2]_{\mathfrak{m}}\big) + B\big([w, h_1]_{\mathfrak{m}}, 0\big) \Rightarrow \qquad \nu|_{\mathfrak{h}\times\mathfrak{h}} = 0 \tag{1.24}$$

$$2B\big(\nu(h, m), w_{\mathfrak{m}}\big) = B\big([w, h]_{\mathfrak{m}}, m\big) = B\big([h, m]_{\mathfrak{m}}, w_{\mathfrak{m}}\big) \Rightarrow \qquad \nu(h, m) = \frac{1}{2}[h, m]_{\mathfrak{m}}, \tag{1.25}$$

where the last transformation follows from the ρ-invariance of B.

Let $u, v \in \mathfrak{m}$, $h \in \mathfrak{h}$, then $B(u_{\mathfrak{m}}, [h, v]_{\mathfrak{m}}) + B([h, u]_{\mathfrak{m}}, v_{\mathfrak{m}}) = -B([h, u_{\mathfrak{m}}], v_{\mathfrak{m}}) + B([h, u_{\mathfrak{m}}], v_{\mathfrak{m}}) = 0$. Thus, the restriction of (1.23) to elements $u, v, w \in \mathfrak{m}$ completely determines $\nu|_{\mathfrak{m}\times\mathfrak{m}}$. With respect to a basis on $\mathfrak{m}$, the equation reduces to a system of linear equations in the coefficients of $\nu|_{\mathfrak{m}\times\mathfrak{m}}$.

In particular, we obtain

$$\Lambda(h).u_{\mathfrak{m}} = \frac{1}{2}[h, u]_{\mathfrak{m}} + \nu(h, u) = \frac{1}{2}\big([h, u]_{\mathfrak{m}} + [h, u]_{\mathfrak{m}}\big) = [h, u]_{\mathfrak{m}} \quad \text{for all } h \in \mathfrak{h},\ u \in \mathfrak{g}.$$

Restricted to fundamental vector fields, (1.3) becomes $\langle \nabla_{\tilde{v}} \tilde{u}, \tilde{w} \rangle + \langle \tilde{v}, \nabla_{\tilde{w}} \tilde{u} \rangle = 0$ for all $u, v, w \in \mathfrak{g}$. Hence, B is $\Lambda(u)$-invariant for all $u \in \mathfrak{g}$, i.e. $B\big(\Lambda(u).v_{\mathfrak{m}}, w_{\mathfrak{m}}\big) + B\big(v_{\mathfrak{m}}, \Lambda(u).w_{\mathfrak{m}}\big) = 0$ for all $v, w \in \mathfrak{g}$.

According to (1.21), the Riemannian curvature in $\mathrm{o} \in G/H$ evaluates as

$$R_{\mathrm{o}}(u_{\mathfrak{m}}, v_{\mathfrak{m}}).w_{\mathfrak{m}} = \Lambda(u).\Lambda(v).w_{\mathfrak{m}} - \Lambda(v).\Lambda(u).w_{\mathfrak{m}} - \Lambda\big([u, v]_{\mathfrak{g}}\big).w_{\mathfrak{m}} \quad \text{for all } u, v, w \in \mathfrak{g}.$$

Thus, all curvature endomorphisms are determined by

$$R_o(u,v) = \big[\Lambda(u),\Lambda(v)\big]_{\mathfrak{gl}} - \Lambda\big([u,v]\big) \qquad \text{for all } u,v \in \mathfrak{m}. \tag{1.26}$$

Algebraically, the three curvature identities in (1.6) follow from (i) the skew symmetry of the definition (1.26), (ii) the Jacobi identity restricted to elements of $\mathfrak{m}$, and (iii) the $\Lambda(u)$-invariance of B for all $u \in \mathfrak{g}$.

The Ricci curvature at $o \in G/H$ is the tensor $\mathrm{Ric}_o : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ determined by

$$\mathrm{Ric}_o(u,v) = \operatorname{tr}\left(w \mapsto R_o(w,u).v\right) \qquad \text{for all } u,v \in \mathfrak{m}.$$

A homogeneous pair $(\mathfrak{g},\mathfrak{h})$ is *reductive*, if there exists a complement $\mathfrak{m}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, so that $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$. With respect to such a decomposition, the adjoint representation Ad restricted to elements of H is of the form

$$\mathrm{Ad}(h) = \begin{array}{cl} \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right) & \begin{array}{l} \mathfrak{h} \\ \mathfrak{m} \end{array} \\ \begin{array}{cc} \mathfrak{h} & \mathfrak{m} \end{array} & \end{array} \qquad \text{where the $*$'s denote invertible matrices that depend on } h \in H.$$

The linear isotropy algebra consists of the mappings $\mathrm{ad}(h)_{\mathfrak{m}} \in \mathfrak{gl}(\mathfrak{m})$ defined by $m \mapsto [h,m]$ for $h \in \mathfrak{h}$, $m \in \mathfrak{m}$, in contrast to $m \mapsto [h,m]_{\mathfrak{m}}$. All homogeneous pairs in this thesis are reductive.

A homogeneous pair $(\mathfrak{g},\mathfrak{h})$ is *symmetric*, if there exists a vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and a Lie algebra automorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ with $\alpha_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$, and $\alpha_{\mathfrak{m}} = -\mathrm{Id}_{\mathfrak{m}}$. Let $h \in \mathfrak{h}$, and $m, m_1, m_2 \in \mathfrak{m}$. The relations

$$\left.\begin{array}{l} [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m} \\ [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h} \end{array}\right\} \quad \text{follow from} \quad \left\{\begin{array}{l} \alpha[h,m] = [\alpha h, \alpha m] = [h,-m] = -[h,m] \\ \alpha[m_1,m_2] = [\alpha m_1, \alpha m_2] = [-m_1,-m_2] = [m_1,m_2]. \end{array}\right.$$

The automorphism α is the starting point to classify Lorenzian symmetric triples, [CW70]. The classification is also carried out in [Ne02].

A homogeneous pair $(\mathfrak{g},\mathfrak{h})$ is *effective*, if $\mathfrak{h}$ contains no non-trivial ideal of $\mathfrak{g}$. An *ideal* of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{j} \subset \mathfrak{g}$ with $[\mathfrak{j},\mathfrak{g}] \subset \mathfrak{j}$. The trivial ideals of $\mathfrak{g}$ are $\{0\}$, and $\mathfrak{g}$. All homogeneous pairs in this thesis are effective.

Two homogeneous pairs $(\mathfrak{g}_1,\mathfrak{h}_1)$, and $(\mathfrak{g}_2,\mathfrak{h}_2)$ are *isomorphic*, if there exists a Lie algebra isomorphism $\alpha : \mathfrak{g}_2 \to \mathfrak{g}_1$ so that $\alpha(\mathfrak{h}_2) = \mathfrak{h}_1$.

A *homogeneous triple* $(\mathfrak{g},\mathfrak{h},B)$ is a homogeneous pair $(\mathfrak{g},\mathfrak{h})$ combined with a ρ-invariant scalar product B on a vector space complement $\mathfrak{m}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. We have illustrated that, locally, a homogeneous triple $(\mathfrak{g},\mathfrak{h},B)$ uniquely determines the corresponding semi-Riemannian homogeneous space G/H with metric $\bar{B}$. Therefore, we shall call two homogeneous triples $(\mathfrak{g}_1,\mathfrak{h}_1,B_1)$, and $(\mathfrak{g}_2,\mathfrak{h}_2,B_2)$ *isomorphic*, if they locally induce the same semi-Riemannian homogeneous space. Equivalently, two homogeneous triples are isomorphic if there exists a Lie algebra isomorphism $\alpha : \mathfrak{g}_2 \to \mathfrak{g}_1$ so that

- $\alpha(\mathfrak{h}_2) = \mathfrak{h}_1$, and
- $B_1(\alpha_{\mathfrak{m}}.u, \alpha_{\mathfrak{m}}.v) = B_2(u,v)$ for all $u,v \in \mathfrak{m}_2$, where $\alpha_{\mathfrak{m}} : \mathfrak{m}_2 \to \mathfrak{m}_1$ denotes the mapping induced by α.

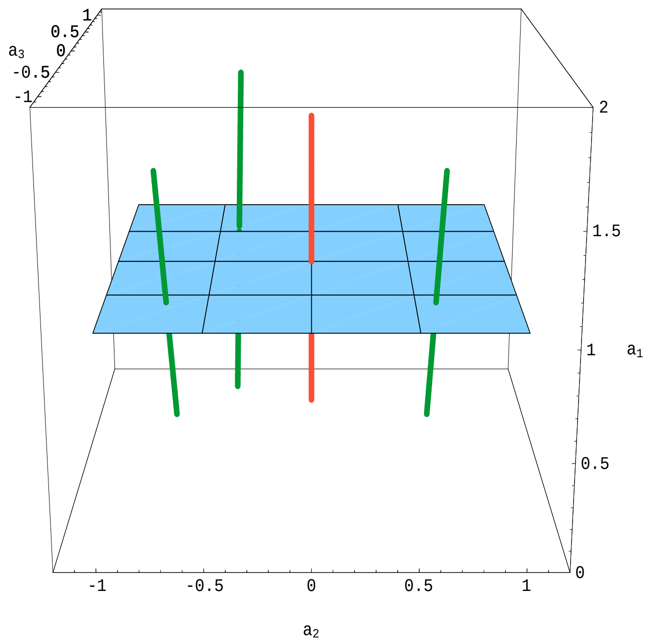

Figure 1.3 : We illustrate V, and M from Example 1.10. Also, we plot sets of the form xV for several $x \in M$.

Remark 1.9. To show that $(\mathfrak{g}_1, \mathfrak{h}_1, B_1)$, and $(\mathfrak{g}_2, \mathfrak{h}_2, B_2)$ are isomorphic, we provide a Lie algebra isomorphism $\alpha : \mathfrak{g}_2 \to \mathfrak{g}_1$ in matrix form as

$$\alpha = \begin{array}{cc} \left(\begin{array}{c|c} C & E \\ \hline 0 & D \end{array}\right) & \begin{array}{c} \mathfrak{h}_1 \\ \mathfrak{m}_1 \end{array} \\ \begin{array}{cc} \mathfrak{h}_2 & \mathfrak{m}_2 \end{array} & \end{array} \qquad \text{with } C \in \mathbb{R}^{\mathrm{h}\times\mathrm{h}}, D \in \mathbb{R}^{\mathrm{m}\times\mathrm{m}} \text{ invertible, and } E \in \mathbb{R}^{\mathrm{h}\times\mathrm{m}},$$

and check the matrix equation $D^T.B_1.D = B_2$. The letters h, m denote the dimensions $\dim \mathfrak{h}_i$, and $\dim \mathfrak{m}_i$, that are the same for $i = 1, 2$. D corresponds to $\alpha_{\mathfrak{m}}$. $\diamond$

Example 1.10. Locally, there exists a diffeomorphism of $\mathrm{SL}(\mathbb{R}^2)$ onto $U \subset \mathbb{R}^3$ with $\mathrm{e} := (1, 0, 0) \in U$, so that the group action of $\mathrm{SL}(\mathbb{R}^2)$ coincides with

$$a \circ b = \left(\frac{a_1 b_1 + a_2 b_3}{a_3 b_2 + 1}, \frac{a_2 + a_1 b_2}{a_3 b_2 + 1}, \frac{a_3 b_1 + b_3}{a_3 b_2 + 1} \right)$$

for points $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ in U close to e. The coordinate e maps $\mathrm{e} \circ a = a$, and $a \circ \mathrm{e} = a$ for all $a \in U$, thus acts as the neutral element. All terms we state in the sequel, are valid only for points sufficiently close to $\mathrm{e} \in U$. For instance, the inverse mapping locally coincides with $\zeta(a) = \frac{1}{a_1}(1, -a_2, -a_3)$.

We have introduced $\mathfrak{sl}(\mathbb{R}^2)$ in Example 1.6. If we choose $X_i := \partial a_i|_{\mathrm{e}}$ as the basis for the Lie algebra $\mathfrak{g} = T_{\mathrm{e}} U \simeq T_{\mathrm{e}} \mathbb{R}^3$, the commutator on elements X_i matches the table in the example. For instance, $[X_2, X_3] = 2X_1$.

We identify $U \subset \mathbb{R}^3 \simeq \mathfrak{g}$. Then, for a vector $X \in \mathfrak{g}$, the image of the exponential mapping $\exp tX = \lambda_X(t) \cdot X$ for $t \in [-\varepsilon, \varepsilon]$ is a straight line in U. However, the function $\lambda_X : [-\varepsilon, \varepsilon] \to \mathbb{R}$ depends on the vector components of X in a non-trivial way.

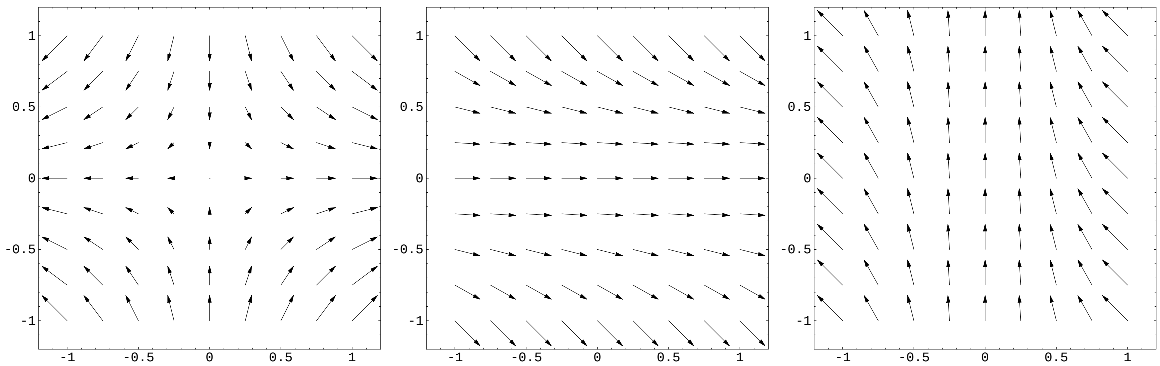

Figure 1.4 : We plot the fundamental vector fields $\tilde{X}_1 = (x_1, -x_2)$, $\tilde{X}_2 = (1, -x_2^2)$, and $\tilde{X}_3 = (-x_1^2, 1)$ on M. Any fundamental vector field $\tilde{u}$ for $u \in \mathfrak{g}$ is just a linear combination of the $\tilde{X}_i$.

The adjoint representation Ad is of the form

$$\mathrm{Ad}(a) = \frac{1}{a_1 - a_2 a_3} \begin{pmatrix} a_1 + a_2 a_3 & -2a_1 a_3 & 2a_2 \\ -a_1 a_2 & a_1^2 & -a_2^2 \\ a_3 & -a_3^2 & 1 \end{pmatrix} \quad \text{for } a \in U. \tag{1.27}$$

We intend to locally model the homogeneous space $\mathrm{SL}(\mathbb{R}^2)/H$, where H is the 1-dimensional closed Lie subgroup $H = \left\{ \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right) : t \in \mathbb{R} \right\} \subset \mathrm{SL}(\mathbb{R}^2)$. The associated homogeneous pair is $\left(\mathfrak{sl}(\mathbb{R}^2), \left\{ \left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right) : t \in \mathbb{R} \right\}\right)$.

Denote with $V = \{(a_1, 0, 0) : a_1 \in \mathbb{R}^*\} \cap U$ the subset of U, which parametrizes H. Naturally, we identify $\mathfrak{h} = \langle X_1 \rangle \subset \mathfrak{g}$. We choose $\mathfrak{m} = \langle X_2, X_3 \rangle$ as the vector space complement, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. The set $M = \{(1, x_1, x_2) : x_i \in \mathbb{R}\} \cap U$ coincides with the image of the exponential map restricted to vectors in $\mathfrak{m}$ close to $0 \in \mathfrak{g}$. Let the x_i for $i = 1, 2$ be the coordinates on M. The point $\mathrm{o} \in M$ has coordinates $(0, 0)$.

For points sufficiently close to $\mathrm{e} \in U$, the projection and the left-action coincide with

$$\pi(a) = \left(a_2, \frac{a_3}{a_1} \right), \quad \text{and} \quad \tau(a, x) = \left(\frac{a_2 + a_1 x_1}{a_3 x_1 + 1}, \frac{a_3 + x_2}{a_1 + a_2 x_2} \right).$$

Elements in the linear isotropy group are of the form $\mathrm{Ad}\big((a_1, 0, 0)\big)_{\mathfrak{m}}$ for $(a_1, 0, 0) \in V$. According to (1.27), we yield

$$\mathrm{Ad}\big((a_1, 0, 0)\big)_{\mathfrak{m}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & \frac{1}{a_1} \end{pmatrix}_{\mathfrak{m}} = \begin{pmatrix} a_1 & 0 \\ 0 & \frac{1}{a_1} \end{pmatrix} \quad \text{for } (a_1, 0, 0) \in V.$$

Since $\mathfrak{h}$ is 1-dimensional, the isotropy representation $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m})$ with $\rho(h) = \mathrm{ad}(h)_{\mathfrak{m}}$ for $h \in \mathfrak{h}$ is determined by the value on the basis element $X_1 \in \mathfrak{h}$. The commutator relations

$$\left. \begin{aligned} [X_1, X_2]_{\mathfrak{m}} &= X_2 \\ [X_1, X_3]_{\mathfrak{m}} &= -X_3 \end{aligned} \right\} \quad \text{define} \quad \rho(X_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The ρ-invariance condition $\rho(X_1)^T.B + B.\rho(X_1) = 0$ of a $(0,2)$-tensor $B : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ on $\mathfrak{m}$ shows that there is 1 degree of freedom, if B is moreover symmetric. The general setup

$$B = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{pmatrix} \qquad \text{results in} \qquad \rho(X_1)^T.B + B.\rho(X_1) = \begin{pmatrix} 2\beta_1 & 0 \\ 0 & -2\beta_3 \end{pmatrix}$$

for coefficients $\beta_i \in \mathbb{R}$. The ρ-invariance demands $\beta_1, \beta_3 = 0$. We substitute $\beta = \beta_2$. B defines a ρ-invariant scalar product on $\mathfrak{m}$ if $\beta \neq 0$. Then, the differential

$$\mathrm{d}\tau_{\zeta(x)}|_{xV} = \frac{1}{1 - x_1 x_2} \begin{pmatrix} \frac{1}{a_1} & 0 \\ 0 & a_1 \end{pmatrix} \qquad \text{extends } B \text{ to} \qquad \bar{B} = \frac{1}{(1 - x_1 x_2)^2} \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix},$$

which defines a left-invariant metric on M. The Riemannian curvature tensor field of $(M, \bar{B})$ is

$$R(\partial x_1, \partial x_2) = \frac{1}{(1 - x_1 x_2)^2} \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \text{while} \qquad \mathrm{Ric} = \frac{-2}{\beta} \bar{B}.$$

Alternatively, we compute $\nu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ as

$$\nu = \begin{array}{c|ccc} & X_1 & X_2 & X_3 \\ \hline X_1 & 0 & \frac{X_2}{2} & -\frac{X_3}{2} \\ X_2 & \frac{X_2}{2} & 0 & 0 \\ X_3 & -\frac{X_3}{2} & 0 & 0 \end{array}$$

to yield the Levi-Civita connection tensor $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ as $\Lambda(X_1) = \rho(X_1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $\Lambda(X_2) = \Lambda(X_3) = 0$. Recall that $\mathfrak{m} = \langle X_2, X_3 \rangle$, so that

$$R_{\mathrm{o}}(X_2, X_3) = \big[\Lambda(X_2), \Lambda(X_3)\big]_{\mathfrak{gl}} - \Lambda\big([X_2, X_3]\big) = -\Lambda\big([X_2, X_3]\big) = -\Lambda(2X_1) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$$

confirms $R(\partial x_1, \partial x_2)_{\mathrm{o}}$, indeed! $\diamond$

Chapter 2

Previous work

The collections of pairwise non-isomorphic homogeneous pairs, and triples in [DK95], and [Ko01] by B. Komrakov, and B. Doubrov make an ideal playground to get acquainted with homogeneous spaces. The classification of 4-dimensional homogeneous pairs in [Ko01] serves as the starting point to detect all 4-dimensional Lorentzian homogeneous triples with isotropy $\dim \mathfrak{h} \geq 1$, and curvature $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$.

To several publications we point later on: M. Cahen and N. Wallach classify the Lorentzian symmetric triples $(\mathfrak{g}, \mathfrak{h}, B)$ with curvature $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$ early in [CW70]. Any such triple is isomorphic to

$$(\mathfrak{g}, \mathfrak{h}, L_{n+2}) \quad \text{with} \quad \mathfrak{g} = \mathfrak{he}_n \rtimes_\delta \mathbb{R}, \quad \text{and} \quad \mathfrak{h} = \langle p_1, \ldots, p_n \rangle .$$

These symmetric triples are covered by our more general class of homogeneous triples, which we introduce in Chapter 4. Corollary 4.12 treats the geometry. In [Ne03], T. Neukirchner reproduces the classification of Lorentzian solvable symmetric triples with curvature $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$.

M. Fels and A. Renner establish the following result in [FR05]: Any 4-dimensional Lorentzian homogeneous triple with isotropy $\dim \mathfrak{h} \geq 1$, and curvature $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$ is reductive. This is in accordance with our more descriptive Corollary 5.11.

2.1 Low-dimensional semi-Riemannian homogeneous spaces

In [DK95], B. Doubrov and B. Komrakov classify the semi-Riemannian homogeneous spaces $(G/H, \bar{B})$ of dimension $\dim G/H \leq 3$. However, they do not cover 3-dimensional Lie groups with left-invariant metric. Their approach is guided by the following result, [DK95] p.3.

Theorem 2.1. *Let $(\mathfrak{g}, \mathfrak{h}, B)$ be a semi-Riemannian homogeneous triple of dimension ≤ 4. There exists a unique semi-Riemannian homogeneous space $(G/H, \bar{B})$ corresponding to $(\mathfrak{g}, \mathfrak{h}, B)$, such that the manifold G/H is simply connected and H is connected.*

First, the authors classify the homogeneous triples $(\mathfrak{g}, \mathfrak{h}, B)$ of dimension ≤ 3. The classification is an exhaustive list of pairwise non-isomorphic homogeneous triples $(\mathfrak{g}, \mathfrak{h}, B)$. The authors omit the case $\dim \mathfrak{g} = 3, \dim \mathfrak{h} = 0$. Secondly, they construct the homogeneous spaces $(G/H, \bar{B})$, corresponding to each homogeneous triple, such that G/H is simply connected and H is connected.

The classification of homogeneous triples is stated without derivation. The description of the homogeneous spaces is sparse. There are gaps in the constructions.

Example 2.2. The homogeneous triples $(\mathfrak{g}, \mathfrak{h}, B)$ of the form

$$\mathfrak{g} = \mathfrak{he}_1 \rtimes_\delta \mathbb{R}, \quad \mathfrak{h} = \langle p_1 \rangle , \quad \text{and} \quad B \text{ has index } 1 \tag{2.1}$$

Table 2.1 : The derivation $\delta : \mathfrak{he}_1 \to \mathfrak{he}_1$ for 3.1, 3.3, 3.4, and 3.5. The real Jordan decomposition $\mathcal{J}(\delta)$ of the matrices δ are shown in the bottom row.

Index	3.1	3.3	3.4	3.5
			$-1 \leq \alpha < 1$	$0 \leq \alpha$
δ	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -\alpha \\ 0 & \alpha+1 & 0 \\ 1 & 0 & \alpha+1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -(\alpha^2+1) \\ 0 & 2\alpha & 0 \\ 1 & 0 & 2\alpha \end{pmatrix}$
$\mathcal{J}(\delta)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha+1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & 2\alpha \end{pmatrix}$

are indexed in the classification of [DK95] as 3.1, 3.3, 3.4, and 3.5. The Lie algebras $\mathfrak{g}$ differ only in the derivation $\delta : \mathfrak{he}_1 \to \mathfrak{he}_1$. Table 2.1 lists the different derivations as (3×3)-matrices with respect to the basis $\mathfrak{he}_1 = \langle p_1, h, q_1 \rangle$. In Corollary 5.8, we confirm that these derivations indeed classify the homogeneous triples of the form (2.1). The Lorentzian scalar product on $\mathfrak{m} = \langle h, q_1, z \rangle$ is $B = L_3$ for all four types 3.1, 3.3, 3.4, and 3.5.

For instance, the homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ with index 3.4 is characterized by

$$\mathrm{ad} = \begin{array}{c|cccc} & p_1 & h & q_1 & z \\ \hline p_1 & 0 & 0 & h & -q_1 \\ h & 0 & 0 & 0 & -(\alpha+1)h \\ q_1 & -h & 0 & 0 & \alpha p_1 - (\alpha+1)q_1 \\ z & q_1 & (\alpha+1)h & -\alpha p_1 + (\alpha+1)q_1 & 0 \end{array}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The pseudo-Riemannian homogeneous spaces $(G/H, \bar{B})$ corresponding to each of the homogeneous triples 3.3, 3.4, 3.5, such that G/H is simply connected and H is connected, share the following properties: The Lie group G is diffeomorphic to $\mathbb{R}^4$. The manifold G/H is diffeomorphic to $\mathbb{R}^3$.

According to the paper, the action $\tau : G \times G/H \to G/H$ varies only slightly among these homogeneous spaces. Unfortunately, the authors provide no derivation for τ. This is fatal, since τ – as stated in the paper – is not well-defined. However, the authors convert each derivation δ to Jordan[1] normalform, which we reproduce in Table 2.1 as $\mathcal{J}(\delta)$. The Jordan normalform seems to be advantageous to design the left-action τ. For this reason, the authors separate 3.3 from 3.4($\alpha = 1$).

[1] Marie Ennemond Camille Jordan, * 5. Jan 1838 in Lyon, † 21. Jan 1922 in Paris

The authors declare the metric $\bar{B}$ on $G/H \simeq \mathbb{R}^3$ as

$$\bar{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & f(x_1, x_2) \end{pmatrix}, \quad \text{defined by}$$

Index	$f(x_1, x_2) =$	
3.3	$-4x_1$	$+x_2^2$
3.4	$-2\alpha(\alpha+1)x_1$	$+\alpha x_2^2$
3.5	$-4\alpha x_1$	$+(2\alpha^2+1)x_2^2$

We have confirmed that the Riemannian curvature R of the semi-Riemannian manifold $(\mathbb{R}^3, \bar{B})$ in $0 \in \mathbb{R}^3$ coincides with R_{o} computed in local terms using (1.26). In fact, Corollary 4.12 states the curvature tensors $R_{\mathrm{o}}, \mathrm{Ric}_{\mathrm{o}}$ for these types of spaces. ◇

2.2 Einstein equation on four-dimensional homogeneous pairs

In [Ko95], B. Komrakov classifies the homogeneous pairs $(\mathfrak{g}, \mathfrak{h})$ of dimension 4 with isotropy $\dim \mathfrak{h} \geq 1$. In [Ko01], he investigates the extentions of the homogeneous *pairs* to homogeneous *triples*. Motivated by the relevance in physics, the author states the solutions to the Einstein-Maxwell[2] equation on each homogeneous pair. In local terms, the Einstein-Maxwell equation is

$$\mathrm{Ric}_{\mathrm{o}} - \lambda B = M_\Omega \qquad \text{with } \lambda \in \mathbb{R}, \tag{2.2}$$

where $B, \mathrm{Ric}_{\mathrm{o}}, M_\Omega$ are symmetric $(0,2)$-tensors on $\mathfrak{m}$. B is a ρ-invariant scalar product on $\mathfrak{m}$. The Ricci curvature $\mathrm{Ric}_{\mathrm{o}}$, and M_Ω are completely determined by the commutator tensor ad on $\mathfrak{g}$ and the scalar product B. In the thesis, we restrict to Lorentzian homogeneous triples $(\mathfrak{g}, \mathfrak{h}, B)$ with solutions to (2.2) subject to

$$\mathrm{Ric}_{\mathrm{o}} = 0, \qquad \text{and} \qquad M_\Omega = 0. \tag{2.3}$$

Then, the Einstein-Maxwell equation reduces to $\mathrm{Ric}_{\mathrm{o}} = \lambda B$. Consult [ON83] pp.336 for the meaning of $M_\Omega \neq 0$ in general relativity theory.

We evince how the material in [Ko01] helps us to detect homogeneous triples with (2.3). In the first chapter of [Ko01] p.42-121, B. Komrakov lists all pairwise non-isomorphic homogeneous pairs $(\mathfrak{g}, \mathfrak{h})$ of dimension 4 with $\dim \mathfrak{h} \geq 1$.

Example 2.3. The homogeneous pair $(\mathfrak{g}, \mathfrak{h})$ indexed as $2.5^2.4$ is part of the classification, [Ko01] p.94. The Lie algebra is $\mathfrak{g} = \mathfrak{he}_2 \rtimes_\delta \mathbb{R}$ with basis $\mathfrak{g} = \langle p_1, p_2, h, q_1, q_2, z \rangle$. The Lie subalgebra is $\mathfrak{h} = \langle p_1, q_2 \rangle$. The derivation $\delta : \mathfrak{he}_2 \to \mathfrak{he}_2$ is the linear mapping defined by the matrix

$$\delta = \begin{pmatrix} 0 & 0 & 0 & s+1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \end{pmatrix}, \qquad 0 \leq s \in \mathbb{R},$$

[2] James Clerk Maxwell, * 13. Jun 1831 in Edinburgh, † 5. Nov 1879 in Cambridge

with respect to $\langle p_1, p_2, h, q_1, q_2\rangle$. Homogeneous pairs with different constants s are non-isomorphic.

Recall from the introduction that a pair $(\mathfrak{g}, \mathfrak{h})$ is characterized by the tensor ad on $\mathfrak{g}$. To avoid any misconception, the author states the commutator on $\mathfrak{g}$ as

$$\mathrm{ad} = \begin{array}{c|cccccc} & e_1 & e_2 & u_1 & u_2 & u_3 & u_4 \\ \hline e_1 & 0 & 0 & 0 & u_1 & -u_2 & 0 \\ e_2 & 0 & 0 & 0 & 0 & u_4 & -u_1 \\ u_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & -u_1 & 0 & 0 & 0 & (s+1)e_1 & 0 \\ u_3 & u_2 & -u_4 & 0 & (-s-1)e_1 & 0 & (1-s)e_2 \\ u_4 & 0 & u_1 & 0 & 0 & (s-1)e_2 & 0 \end{array} \qquad \text{for } 0 \leq s \in \mathbb{R}.$$

The table matches the commutator of $\mathfrak{h}\mathfrak{c}_2 \rtimes_\delta \mathbb{R}$ if we set $e_1 = q_2$, $e_2 = p_1$, $u_1 = -h$, $u_2 = p_2$, $u_3 = z$, and $u_4 = q_1$. $\diamond$

In the second chapter, the author states all possible solutions to the Einstein-Maxwell equation on each of the 4-dimensional homogeneous pairs $(\mathfrak{g}, \mathfrak{h})$. The solutions are given in terms of B, λ, Ω. The skew symmetric (1,1)-tensor Ω on $\mathfrak{m}$ relates to the M_Ω as

$$M_\Omega = -\Omega^T.B^{-1}.\Omega. \tag{2.4}$$

In case B is of index 1, Lemma 1.3 warrants the equivalence $M_\Omega = 0 \Leftrightarrow \Omega = 0$. Thus, no solution to the Einstein-Maxwell equation B, λ, Ω with $\Omega \neq 0$ leads to a homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ subject to (2.3).

Example 2.4. We continue the discussion of the homogeneous pair from the previous example. Any solution to the Einstein-Maxwell equation in terms of B, λ, Ω is of the form, [Ko01] p.156,

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \lambda = 0,$$

where $a, b, \alpha, \beta \in \mathbb{R}$ satisfy the relations $2a = -(\alpha^2 + \beta^2)$, and $\alpha^2 + \beta^2 \neq 0$.

Substituting $\lambda = 0$ in the Einstein-Maxwell equation, we obtain $\mathrm{Ric}_o = M_\Omega$, where

$$M_\Omega = -\Omega^T.B^{-1}.\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(\alpha^2+\beta^2)/a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

However, setting $\Omega = 0$ necessarily annihilates α, β, which violates the condition $\alpha^2 + \beta^2 \neq 0$. Thus, the pair $(\mathfrak{g}, \mathfrak{h})$ does not extend to a homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ subject to (2.3). $\diamond$

Chapter 3

Lorentzian homogeneous triples from Komrakov's list

We are interested in homogeneous triples $(\mathfrak{g}, \mathfrak{h}, B)$ with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, that have certain geometric properties:

$$\left.\begin{array}{cl} B \text{ has index } 1 & \text{the scalar product } B \text{ on } \mathfrak{m} \text{ is Lorentzian} \\ R_{\mathrm{o}} \neq 0 & \text{not Riemannian-flat} \\ \mathrm{Ric}_{\mathrm{o}} = 0 & \text{Ricci-flat} \end{array}\right\} \tag{3.1}$$

If $\dim \mathfrak{m} \leq 3$, no such triples can exist, since for semi-Riemannian manifolds of dimension ≤ 3 the implication $\mathrm{Ric} = 0 \Rightarrow R = 0$ holds. For instance, consider a triple of dimension 3. A $(1,3)$-tensor R_{o} on $\mathfrak{m}$ that satisfies the curvature identities (1.6) locally has 6 degrees of freedom. Let B be a Lorentzian scalar product on $\mathfrak{m}$. We choose a basis on $\mathfrak{m} = \langle u_1, u_2, u_3 \rangle$, so that $B = L_3$. Then, the curvature R_{o}, and $\mathrm{Ric}_{\mathrm{o}}$ are of the form

$$R_{\mathrm{o}}(u_1,u_2) = \begin{pmatrix} -c_4 & -c_2 & 0 \\ -c_1 & 0 & c_2 \\ 0 & c_1 & c_4 \end{pmatrix} \quad R_{\mathrm{o}}(u_1,u_3) = \begin{pmatrix} -c_6 & -c_5 & 0 \\ -c_4 & 0 & c_5 \\ 0 & c_4 & c_6 \end{pmatrix} \quad R_{\mathrm{o}}(u_2,u_3) = \begin{pmatrix} -c_5 & -c_3 & 0 \\ -c_2 & 0 & c_3 \\ 0 & c_2 & c_5 \end{pmatrix} \quad \Bigg| \quad \mathrm{Ric}_{\mathrm{o}} = \begin{pmatrix} c_1 & -c_4 & -c_2 - c_6 \\ -c_4 & -2c_2 & -c_5 \\ -c_2 - c_6 & -c_5 & c_3 \end{pmatrix}$$

with coefficients $c_i \in \mathbb{R}$ for $i = 1, \dots, 6$. We have $\mathrm{Ric}_{\mathrm{o}} = 0 \Rightarrow R_{\mathrm{o}} = 0$, because demanding $\mathrm{Ric}_{\mathrm{o}} = 0$ eliminates all coefficients c_i.

In [Ko01], B. Komrakov publishes a list of all pairwise non-isomorphic 4-dimensional homogeneous pairs $(\mathfrak{g}, \mathfrak{h})$ with $\dim \mathfrak{h} \geq 1$. In addition, the author references those pairs, which admit a ρ-invariant Lorentzian scalar product on $\mathfrak{m}$. According to Theorem II.2.2 [Ko01] p.164, there are 63 such pairs.

In the next section, we argue that 10 of the 63 homogeneous *pairs* in question extend to homogeneous *triples* $(\mathfrak{g}, \mathfrak{h}, B)$ of the desired geometry (3.1). Suppose the classification of 4-dimensional homogeneous pairs in [Ko01] is correct. Then, a sequence of lemmas in Section 3.2 proves that any 4-dimensional Lorentzian homogeneous triple with curvature $R_{\mathrm{o}} \neq 0$, and $\mathrm{Ric}_{\mathrm{o}} = 0$ is isomorphic to

$$(\mathfrak{g}, \mathfrak{h}, L_4) \quad \text{with} \quad \mathfrak{g} = \mathfrak{he}_n \times \mathbb{R}^m \rtimes_\delta \mathbb{R}, \quad \text{and} \quad \mathfrak{h} = \langle p_1, \dots, p_n \rangle$$

for either $n = 1, m = 1$, or $n = 2, m = 0$.

3.1 Selection of homogeneous pairs

Section 2.2 provides a concise overview on B. Komrakov's work [Ko01]. A homogeneous pair in his classification is explicitly characterized by the commutator tensor ad. Additionally, for each pair the author states all possible solutions to the Einstein-Maxwell equation

$$\mathrm{Ric}_{\mathrm{o}} - \lambda B = M_\Omega \quad \text{ with } \lambda \in \mathbb{R}, \tag{3.2}$$

in terms of B, λ, Ω, and $M_\Omega = -\Omega^T.B^{-1}.\Omega$. Having ad, B, Ω, λ at hand, there are simple criteria for a pair $(\mathfrak{g}, \mathfrak{h})$ not to extend to a triple $(\mathfrak{g}, \mathfrak{h}, B)$ with geometric properties (3.1).

According to Theorem II.2.2 [Ko01] p.164, 63 homogeneous pairs extend to *Lorentzian* homogeneous triples. However, we show that 47 of these 63 triples do not satisfy $R_{\mathrm{o}} \neq 0$, or $\mathrm{Ric}_{\mathrm{o}} = 0$ by applying one of the arguments below.

- $\boxed{[\mathfrak{m}, \mathfrak{m}] = 0}$ Any homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] = \{0\}$ is Riemannian-flat, i.e. $R_{\mathrm{o}} = 0$. Lemma 3.1 below gives the proof. $[\mathfrak{m}, \mathfrak{m}] = \{0\}$ is determined easily from the commutator tensor ad.

- $\boxed{\lambda \neq 0}$ Non-zero λ in the Einstein-Maxwell equation $\mathrm{Ric}_{\mathrm{o}} - \lambda B = M_\Omega$ causes either $\mathrm{Ric}_{\mathrm{o}} \neq 0$, or $M_\Omega \neq 0$. However, we demand $\mathrm{Ric}_{\mathrm{o}} = 0$, and $M_\Omega = 0$.

- $\boxed{|B| = 0}$ The entries of Ω and B sometimes are correlated. Setting $\Omega = 0$ might cause the determinant $|B| = 0$ to vanish. Then, B does not define a scalar product on $\mathfrak{m}$.

- $\boxed{|B| > 0}$ Similarly, the correlation between Ω and B might enforce $|B| > 0$. But, the determinant of the matrix associated to a Lorentzian scalar product is always negative.

- $\boxed{R_{\mathrm{o}} = 0}$ This label refers to pairs whose Lorentzian, Ricci-flat extensions are necessarily Riemannian-flat. The Riemannian curvature R_{o} is not stated explicitly in [Ko01].

Table 3.1 minutes the complete selection process. Due to the extent, we do not derive the properties of all 63 pairs in this thesis. However, we demonstrate the application of each criteria in an adequate example. Ambitious readers, who are in possession of a copy of [Ko01] may easily verify the selection process using Table 3.1.

Lemma 3.1. Any homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] = \{0\}$ is Riemannian-flat, i.e. $R_{\mathrm{o}} = 0$.

Proof. The Riemannian curvature tensor R_{o} of a triple $(\mathfrak{g}, \mathfrak{h}, B)$ involves the computation of the terms ν, Λ. First, we show that $[\mathfrak{m}, \mathfrak{m}] = \{0\} \Rightarrow \nu(u, v) = 0$ for all $u, v \in \mathfrak{m}$. The implication is true, since $\nu|_{\mathfrak{m}\times\mathfrak{m}}$ is determined by

$$\begin{aligned} 2B(\nu(u,v), w) &= B(u, [w,v]_{\mathfrak{m}}) + B([w,u]_{\mathfrak{m}}, v) \quad \text{ for all } \underline{u, v, w \in \mathfrak{m}} \\ &= B(u, 0) + B(0, v) = 0. \end{aligned}$$

To obtain $R_{\mathrm{o}}(u, v) = \big[\Lambda(u), \Lambda(v)\big]_{\mathfrak{gl}} - \Lambda\big([u, v]\big) = 0$ for all $u, v \in \mathfrak{m}$, it clearly suffices to show $\Lambda(u) = 0$ for all $u \in \mathfrak{m}$. Note, $\Lambda(u).v = \frac{1}{2}[u, v]_{\mathfrak{m}} + \nu(u, v) = 0$ for all $u, v \in \mathfrak{m}$. $\square$

Table 3.1 : Indices of homogeneous pairs that admit a ρ-invariant Lorentzian scalar product on $\mathfrak{m}$ according to Theorem II.2.2 [Ko01] p.164. A mark in the table indicates which selection criteria applies to a homogeneous pair. A tuple (i, j) refers to Example $i.j$ in the thesis.

Index	$[\mathfrak{h}, \mathfrak{m}] \not\subset \mathfrak{m}$	$[\mathfrak{m}\mathfrak{m}] = 0$	$\lambda \neq 0$	$\lvert B\rvert = 0$	$\lvert B\rvert > 0$	$R_{\mathrm{o}} = 0$	fine
$1.1^1.2$					•		
$1.1^1.5$				• (3.4)			
$1.1^1.6$					•		
$1.1^1.7$				•			
$1.1^1.10$		•					
$1.1^2.2$						• (3.7)	
$1.1^2.6$				•			
$1.1^2.7$				•			
$1.1^2.8$						• (3.7)	
$1.1^2.9$				•			
$1.1^2.10$				•			
$1.1^2.12$		•					
$1.1^3.1$		•					
$1.1^4.1$		•					
$1.4^1.2$	•					• (3.7)	
$1.4^1.3$	•		• (3.3)				
$1.4^1.4$	•		•				
$1.4^1.8$			•				
$1.4^1.9$							•
$1.4^1.10$							•
$1.4^1.11$							•
$1.4^1.12$						• (5.3)	
$1.4^1.13$							•
$1.4^1.14$							•
$1.4^1.15$					• (3.5)		
$1.4^1.16$							•
$1.4^1.17$				•			
$1.4^1.18$					•		
$1.4^1.19$							•
$1.4^1.20$				•			
$1.4^1.21$				•			
$1.4^1.22$				•			
$1.4^1.23$						• (5.3)	
$1.4^1.24$				•			
$1.4^1.25$				•			
$1.4^1.26$		• (3.2)					
$2.1^2.1$			•				
$2.1^2.2$			•				
$2.1^2.3$			•				
$2.1^2.4$				•			
$2.1^2.5$				•			
$2.1^2.6$		•					
$2.4^1.2$			•				
$2.4^1.3$		•					
$2.5^2.1$	•		•				

Table 3.1 (continued)

index	$[\mathfrak{h},\mathfrak{m}] \not\subset \mathfrak{m}$	$[\mathfrak{m}\mathfrak{m}]=0$	$\lambda \neq 0$	$\lvert B\rvert = 0$	$\lvert B\rvert > 0$	$R_o = 0$	fine
$2.5^2.2$							•
$2.5^2.3$							•
$2.5^2.4$				• (2.4)			
$2.5^2.5$				•			
$2.5^2.6$							•
$2.5^2.7$		•					
$3.2^2.1$	•		•				
$3.2^2.2$		•					
$3.3^2.1$						• (3.6)	
$3.3^2.4$		•					
$3.5^1.1$			•				
$3.5^1.4$		•					
$3.5^2.1$			•				
$3.5^2.4$		•					
$4.1^2.1$		•					
$6.1^3.1$			•				
$6.1^3.2$			•				
$6.1^3.3$		•					

Within the following examples, we denote the basis elements of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ by $\mathfrak{h} = \langle e_1, \ldots, e_{\dim \mathfrak{h}} \rangle$ and $\mathfrak{m} = \langle u_1, \ldots, u_4 \rangle$.

Example 3.2. $\boxed{[\mathfrak{m},\mathfrak{m}] = 0}$ In [Ko01] p.73, the homogeneous pair $(\mathfrak{g}, \mathfrak{h})$ with index $1.4^1.26$ is defined by

$$\mathrm{ad} = \begin{array}{c|ccccc} & e_1 & u_1 & u_2 & u_3 & u_4 \\ \hline e_1 & 0 & 0 & u_1 & u_2 & 0 \\ u_1 & 0 & 0 & 0 & 0 & 0 \\ u_2 & -u_1 & 0 & 0 & 0 & 0 \\ u_3 & -u_2 & 0 & 0 & 0 & 0 \\ u_4 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

We notice $[\mathfrak{m},\mathfrak{m}] = 0$, thus $R_o = 0$ by the previous lemma. ◇

Example 3.3. $\boxed{\lambda \neq 0}$ Proposition $1.4^1.3$ in [Ko01] p.143 states that any solution of the Einstein-Maxwell equation on the pair $(\mathfrak{g}, \mathfrak{h})$ defined by

$$\mathrm{ad} = \begin{array}{c|ccccc} & e_1 & u_1 & u_2 & u_3 & u_4 \\ \hline e_1 & 0 & 0 & u_1 & u_2 & e_1 \\ u_1 & 0 & 0 & 0 & 0 & 2u_1 \\ u_2 & -u_1 & 0 & 0 & e_1 & u_2 \\ u_3 & -u_2 & 0 & -e_1 & 0 & 0 \\ u_4 & -e_1 & -2u_1 & -u_2 & 0 & 0 \end{array} \quad \text{has the form} \quad \left\{ \begin{array}{rl} B = & \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & d & c \\ 0 & 0 & c & d \end{pmatrix}, \\ \Omega = & 0, \\ \lambda = & -\frac{3}{d} \end{array} \right.$$

with coefficients $a, c, d \in \mathbb{R}$ such that $\det B \neq 0$.

Since $\lambda = -\frac{3}{d} \neq 0$, we either have $\mathrm{Ric}_o \neq 0$, or $M_\Omega \neq 0$ by the Einstein-Maxwell equation. In this particular case, $M_\Omega = -\Omega^T.B^{-1}.\Omega = 0$, but $\mathrm{Ric}_o = \lambda B \neq 0$. $\diamond$

Example 3.4. $\boxed{|B| = 0}$ In [Ko01] p.43, the homogeneous pair $(\mathfrak{g}, \mathfrak{h})$ with index $1.1^1.5$ is defined by

$$\mathrm{ad} = \begin{array}{c|ccccc} & e_1 & u_1 & u_2 & u_3 & u_4 \\ \hline e_1 & 0 & u_1 & 0 & -u_3 & 0 \\ u_1 & -u_1 & 0 & 0 & e_1 & 0 \\ u_2 & 0 & 0 & 0 & 0 & u_2 \\ u_3 & u_3 & -e_1 & 0 & 0 & 0 \\ u_4 & 0 & 0 & -u_2 & 0 & 0 \end{array}.$$

Proposition $1.1^1.5$, [Ko01] p.128, states that any solution of the Einstein-Maxwell equation on $(\mathfrak{g}, \mathfrak{h})$ has the form:

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \lambda = \frac{\beta^2 - b}{bd - c^2},$$

where

$$\alpha^2 = \frac{b - \beta^2}{bd - c^2} a^2 - a. \tag{3.3}$$

Setting $\alpha, \beta = 0$ in Ω, simplifies $\lambda = -\frac{b}{bd-c^2}$. We require $\lambda = 0$, which we enforce by $b = 0$. But then relation (3.3) reduces to $a = 0$, which kills the determinant of B. $\diamond$

Example 3.5. $\boxed{|B| > 0}$ In [Ko01] p.66, the homogeneous pair $(\mathfrak{g}, \mathfrak{h})$ with index $1.4^1.15$ is defined by

$$\mathrm{ad} = \begin{array}{c|ccccc} & e_1 & u_1 & u_2 & u_3 & u_4 \\ \hline e_1 & 0 & 0 & u_1 & u_2 & 0 \\ u_1 & 0 & 0 & 0 & 0 & 0 \\ u_2 & -u_1 & 0 & 0 & e_1 + u_4 & 0 \\ u_3 & -u_2 & 0 & -e_1 - u_4 & 0 & u_1 \\ u_4 & 0 & 0 & 0 & -u_1 & 0 \end{array}.$$

Proposition $1.4^1.15$, [Ko01] p.144, states that any solution of the Einstein-Maxwell equation on $(\mathfrak{g}, \mathfrak{h})$ has the form:

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \lambda = 0, \qquad \text{where} \quad 1 + \frac{d}{2a} = \frac{\alpha^2}{a} + \frac{\beta^2}{d}.$$

Setting $\alpha, \beta = 0$ in Ω, we retrieve the relation $d = -2a$, but then the determinant $|B| = -a^3 d = 2a^4$ is positive. Such a matrix B does not define a Lorentzian scalar product. $\diamond$

Example 3.6. $\boxed{R_\circ = 0}$ In [Ko01] p.102, the homogeneous pair $(\mathfrak{g}, \mathfrak{h})$ with index $3.3^2.1$ is defined by

$$\mathrm{ad} = \begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & u_1 & u_2 & u_3 & u_4 \\ \hline e_1 & 0 & -e_3 & e_2 & 0 & u_4 & 0 & -u_2 \\ e_2 & e_3 & 0 & 0 & 0 & u_1 & -u_2 & 0 \\ e_3 & -e_2 & 0 & 0 & 0 & 0 & u_4 & -u_1 \\ u_1 & 0 & 0 & 0 & 0 & 0 & u_1 & 0 \\ u_2 & -u_4 & -u_1 & 0 & 0 & 0 & pe_2 + u_2 & 0 \\ u_3 & 0 & u_2 & -u_4 & -u_1 & -pe_2 - u_2 & 0 & pe_3 - u_4 \\ u_4 & u_2 & 0 & u_1 & 0 & 0 & u_4 - pe_3 & 0 \end{array} \qquad \text{with } p \in \mathbb{R}.$$

Proposition $3.3^2.1$, [Ko01] p.159, states that any solution of the Einstein-Maxwell equation on $(\mathfrak{g}, \mathfrak{h})$ has the form:

$$p = 0, \quad B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \lambda = 0.$$

The determinant $|B| = -a^4$. If $a > 0$, $(\mathfrak{g}, \mathfrak{h}, B)$ defines a Lorentzian homogeneous triple. For $\nu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ on pairs of elements from the basis $\mathfrak{g} = \langle e_1, e_2, e_3, u_1, u_2, u_3, u_4 \rangle$ we yield

$$2\nu = \begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & u_1 & u_2 & u_3 & u_4 \\ \hline e_1 & 0 & 0 & 0 & 0 & u_4 & 0 & -u_2 \\ e_2 & 0 & 0 & 0 & 0 & u_1 & -u_2 & 0 \\ e_3 & 0 & 0 & 0 & 0 & 0 & u_4 & -u_1 \\ u_1 & 0 & 0 & 0 & 0 & 0 & -u_1 & 0 \\ u_2 & u_4 & u_1 & 0 & 0 & -2u_1 & u_2 & 0 \\ u_3 & 0 & -u_2 & u_4 & -u_1 & u_2 & 2\left(u_3 - \frac{bu_1}{a}\right) & u_4 \\ u_4 & -u_2 & 0 & -u_1 & 0 & 0 & u_4 & -2u_1 \end{array}.$$

The reader verifies easily the relations $\nu(u_\mathfrak{h}, v_\mathfrak{h}) = 0$, and $\nu(u_\mathfrak{h}, v_\mathfrak{m}) = \frac{1}{2}[u_\mathfrak{h}, v_\mathfrak{m}]$ for $u, v \in \mathfrak{g}$. The Levi-Civita connection tensor $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ is determined on basis elements as

$$\Lambda(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \Lambda(e_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Lambda(e_3) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$\Lambda(u_1) = 0,$

$$\Lambda(u_2) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Lambda(u_3) = \begin{pmatrix} -1 & 0 & -\frac{b}{a} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Lambda(u_4) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The Riemannian curvature $R_\mathrm{o} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ is defined as $R_\mathrm{o}(u,v) = \big[\Lambda(u), \Lambda(v)\big]_{\mathfrak{gl}} - \Lambda\big([u,v]\big)$ for all $u, v \in \mathfrak{m}$. To see that $R_\mathrm{o} = 0$, we simply check $\big[\Lambda(u), \Lambda(v)\big]_{\mathfrak{gl}} = \Lambda\big([u,v]\big)$ for all $u, v \in \mathfrak{m}$. Since $p = 0$, the commutator tensor $\mathrm{ad}|_{\mathfrak{m}\times\mathfrak{m}}$ simplifies to

$[,]\vert_{\mathfrak{m}\times\mathfrak{m}}$	u_1	u_2	u_3	u_4
u_1	0	0	u_1	0
u_2	0	0	u_2	0
u_3	$-u_1$	$-u_2$	0	$-u_4$
u_4	0	0	u_4	0

. Indeed,

$[,]_{\mathfrak{gl}}$	$\Lambda(u_1)$	$\Lambda(u_2)$	$\Lambda(u_3)$	$\Lambda(u_4)$
$\Lambda(u_1)$	0	0	$\Lambda(u_1)$	0
$\Lambda(u_2)$	0	0	$\Lambda(u_2)$	0
$\Lambda(u_3)$	$-\Lambda(u_1)$	$-\Lambda(u_2)$	0	$-\Lambda(u_4)$
$\Lambda(u_4)$	0	0	$\Lambda(u_4)$	0

.

◇

Example 3.7. We treat the homogeneous pairs with indices $1.1^2.2$, $1.1^2.8$, and $1.4^1.2$ on the Internet page [Ha06]. Assuming a general ρ-invariant Lorentzian scalar product, we display the corresponding geometric tensors $\nu, \Lambda, R_\mathrm{o}, \mathrm{Ric}_\mathrm{o}$. It turns out, that $\mathrm{Ric}_\mathrm{o} = 0 \Rightarrow R_\mathrm{o} = 0$. ◇

3.2 Four-dimensional Lorentzian Ricci-flat homogeneous triples

The next lemmas summarize the 10 four-dimensional homogeneous pairs $(\mathfrak{g}, \mathfrak{h})$ in [Ko01] that extend to homogeneous triples $(\mathfrak{g}, \mathfrak{h}, B)$ so that

$$\left.\begin{array}{ll} B \text{ has index } 1 & \text{the scalar product } B \text{ on } \mathfrak{m} \text{ is Lorentzian} \\ R_{\mathrm{o}} \neq 0 & \text{not Riemannian-flat} \\ \mathrm{Ric}_{\mathrm{o}} = 0 & \text{Ricci-flat} \end{array}\right\} \tag{3.4}$$

In the next chapter, we derive the geometry of these homogeneous triples in a more general context. We ommit the proofs of the lemmas at this point.

Lemma 3.8. $\boxed{\dim \mathfrak{h} = 1}$ Let $(\mathfrak{g}, \mathfrak{h})$ be a homogeneous pair with $\dim \mathfrak{h} = 1$ from [Ko01]. Suppose $(\mathfrak{g}, \mathfrak{h})$ extends to a homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ with properties (3.4). Then, $(\mathfrak{g}, \mathfrak{h}, B)$ is of isomorphy type $\mathfrak{g} = \mathfrak{he}_1 \times \mathbb{R} \rtimes_\delta \mathbb{R}$ with $\mathfrak{g} = \langle p_1, h, q_1, r_1, z \rangle$, and isotropy $\mathfrak{h} = \langle p_1 \rangle$. The ρ-invariant Lorentzian scalar product on $\mathfrak{m} = \langle h, q_1, r_1, z \rangle$ is

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ a & 0 & c & d \end{pmatrix} \qquad \text{with } a, b, c, d \in \mathbb{R} \text{ and } a, b > 0. \tag{3.5}$$

The derivation $\delta : \mathfrak{he}_1 \times \mathbb{R} \to \mathfrak{he}_1 \times \mathbb{R}$ with respect to $\langle p_1, h, q_1, r_1 \rangle$ is from the following list:

Index	$1.4^1.9$	$1.4^1.10$	$1.4^1.11$	$1.4^1.13$
δ	$\begin{pmatrix} 0 & 0 & r & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -p \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & r & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & r & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$
Index	$1.4^1.14$	$1.4^1.16$	$1.4^1.19$	
δ	$\begin{pmatrix} 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	

The parameters are $r, p \in \mathbb{R}$ with restrictions as indicated by the table below.

Index	$\mathrm{Ric}_{\mathrm{o}} = 0 \Rightarrow$	$R_{\mathrm{o}} \neq 0 \Rightarrow$
$1.4^1.9$	$r = \frac{-2ap^2-2ap-b}{2a}$	$p \neq -\frac{1}{2} \vee -4ap^2 - 4ap + b \neq 0$
$1.4^1.10$	$r = -p^2 - p$	$p(p+1) \neq 0$
$1.4^1.11$	$r = -\frac{b}{2a}$	
$1.4^1.13$	$r = \frac{-2a-b}{2a}$	
$1.4^1.14$	$r = -1$	

Lemma 3.9. $\boxed{\dim\mathfrak{h}=2}$ Let $(\mathfrak{g},\mathfrak{h})$ be a homogeneous pair with $\dim\mathfrak{h}=2$ from [Ko01]. Suppose $(\mathfrak{g},\mathfrak{h})$ extends to a homogeneous triple $(\mathfrak{g},\mathfrak{h},B)$ with properties (3.4). Then, $(\mathfrak{g},\mathfrak{h},B)$ is of isomorphy type $\mathfrak{g}=\mathfrak{he}_2\rtimes_\delta\mathbb{R}$ with $\mathfrak{g}=\langle p_1,p_2,h,q_1,q_2,z\rangle$, and isotropy $\mathfrak{h}=\langle p_1,p_2\rangle$. The ρ-invariant Lorentzian scalar product on $\mathfrak{m}=\langle h,q_1,q_2,z\rangle$ is

$$B=\begin{pmatrix}0&0&0&a\\0&a&0&0\\0&0&a&0\\a&0&0&b\end{pmatrix}\qquad\text{with } a,b\in\mathbb{R} \text{ and } a>0. \tag{3.6}$$

The derivation $\delta:\mathfrak{he}_2\to\mathfrak{he}_2$ is from the following list:

#	$2.5^2.2$	$2.5^2.3$	$2.5^2.6$
δ	$\begin{pmatrix}0&2t&0&-4t^2-r+s&-t\\0&0&0&-t&-r-s\\0&0&-1&0&0\\1&0&0&-1&0\\0&1&0&-2t&-1\end{pmatrix}$	$\begin{pmatrix}0&1&0&-r-s-1&0\\0&0&0&0&s-r\\0&0&0&0&0\\1&0&0&0&0\\0&1&0&-1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}$

We state δ with respect to the basis $\mathfrak{he}_2=\langle p_1,p_2,h,q_1,q_1\rangle$. The parameters are $r,s,t\in\mathbb{R}$ with $s,t\geq 0$ and restrictions as indicated by the table below.

Index	$\mathrm{Ric}_o=0\Rightarrow$	$R_o\neq 0\Rightarrow$
$2.5^2.2$	$r=-t^2$	$s>0$
$2.5^2.3$	$r=-\frac{1}{4}$	$s>0$

■

B. Komrakov does not explicitly state the domains of the coefficients $a,b,c,d\in\mathbb{R}$, for which B makes a scalar product of index 1. For that reason, we have derived the domains in Example 1.1. The homogeneous pair with index $2.5^2.6$ is a symmetric pair, which is part of the classification of solvable pseudo-Riemannian symmetric spaces of index 1 and 2 in [Ne03] p.31.

In accordance with Table 3.1, the previous lemmas 3.8 and 3.9 cover all homogeneous pairs from [Ko01], that extend to homogeneous triples of the desired geometry (3.4). Suppose Komrakov's classification is correct. Then, no other 4-dimensional homogeneous triples $(\mathfrak{g},\mathfrak{h},B)$ with $\dim\mathfrak{h}\geq 1$, properties (3.4), and essentially different Lie algebra structure $\mathfrak{g}$ exist.

In the following lemmas, we establish important isomorphies. We show that fixing $B=L_4$ is not a restriction.

Lemma 3.10. $\boxed{\dim\mathfrak{h}=1}$ Any homogeneous triple $(\mathfrak{g},\mathfrak{h},B)$ encountered in Lemma 3.8 is isomorphic to $(\breve{\mathfrak{g}},\breve{\mathfrak{h}},L_4)$ with $\breve{\mathfrak{g}}=\mathfrak{he}_1\times\mathbb{R}\rtimes_{\breve{\delta}}\mathbb{R}$, $\breve{\mathfrak{h}}=\langle p_1\rangle$, and $\breve{\delta}:\mathfrak{he}_1\times\mathbb{R}\to\mathfrak{he}_1\times\mathbb{R}$ is a derivation.

Proof. According to the table of derivations in Lemma 3.8, we may assume δ with respect to

$\mathfrak{he}_1 \times \mathbb{R} = \langle p_1, h, q_1, r_1 \rangle$ is of the form

$$\delta = \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & x & 0 & g \\ 1 & 0 & x & 0 \\ 0 & 0 & n & p \end{pmatrix}, \qquad x, s, n, p, g \in \mathbb{R}. \qquad \text{Then,} \quad \breve{\delta} = \begin{pmatrix} 0 & 0 & \frac{s}{a} & 0 \\ 0 & \frac{x}{\sqrt{a}} & 0 & \frac{g}{\sqrt{b}} \\ 1 & 0 & \frac{x}{\sqrt{a}} & 0 \\ 0 & 0 & \frac{\sqrt{b}n}{a} & \frac{p}{\sqrt{a}} \end{pmatrix},$$

where $a, b > 0$ are the coefficients of the Lorentzian scalar product B in (3.5). To see this, denote with $\breve{\mathrm{ad}}$ the commutator of $\breve{\mathfrak{g}} = \mathfrak{he}_1 \times \mathbb{R} \rtimes_{\breve{\delta}} \mathbb{R}$, and ad is the commutator of $\mathfrak{g}$. In the spirit of Remark 1.9, we provide a Lie algebra isomorphism $\alpha : \breve{\mathfrak{g}} \to \mathfrak{g}$ such that the diagrams

$$\begin{array}{cccccc} \breve{\mathrm{ad}}: & \breve{\mathfrak{g}} & \times & \breve{\mathfrak{g}} & \to & \breve{\mathfrak{g}} \\ & \downarrow \alpha & & \downarrow \alpha & & \uparrow \alpha^{-1} \\ \mathrm{ad}: & \mathfrak{g} & \times & \mathfrak{g} & \to & \mathfrak{g} \end{array}, \qquad \text{and} \qquad \begin{array}{cccccc} L_4: & \breve{\mathfrak{m}} & \times & \breve{\mathfrak{m}} & \to & \mathbb{R} \\ & \downarrow \alpha_{\mathfrak{m}} & & \downarrow \alpha_{\mathfrak{m}} & & \uparrow \mathrm{Id} \\ B: & \mathfrak{m} & \times & \mathfrak{m} & \to & \mathbb{R} \end{array}$$

commute. We choose $\alpha = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \alpha_{\mathfrak{m}} \end{array}\right)$, where $\alpha_{\mathfrak{m}}$ is adapted from Example 1.1 as

$$\alpha_{\mathfrak{m}} = \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 & 0 & \frac{c^2-bd}{2a^{3/2}b} \\ 0 & \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{b}} & -\frac{c}{\sqrt{ab}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{a}} \end{pmatrix} \quad \text{with inverse} \quad \alpha_{\mathfrak{m}}^{-1} = \begin{pmatrix} \sqrt{a} & 0 & 0 & \frac{bd-c^2}{2\sqrt{ab}} \\ 0 & \sqrt{a} & 0 & 0 \\ 0 & 0 & \sqrt{b} & \frac{c}{\sqrt{b}} \\ 0 & 0 & 0 & \sqrt{a} \end{pmatrix}.$$

The matrix product $\alpha_{\mathfrak{m}}^T.B.\alpha_{\mathfrak{m}} = L_4$ proves the right diagram. Next, we show the commutativity of the left diagram. Fix the basis $\breve{\mathfrak{g}} = \left\langle \breve{p}_1, \breve{h}, \breve{q}_1, \breve{r}_1, \breve{z} \right\rangle$. Due to the diagonal shape of α restricted to the first four columns, the only non-zero commutator relation of $\breve{\mathrm{ad}}$ on elements of $\left\langle \breve{p}_1, \breve{h}, \breve{q}_1, \breve{r}_1 \right\rangle$ is

$$[\breve{p}_1, \breve{q}_1]_{\breve{\mathfrak{g}}} = \alpha^{-1} \left[\alpha\breve{p}_1, \alpha\breve{q}_1\right]_{\mathfrak{g}} = \alpha^{-1} \left[1p_1, \frac{1}{\sqrt{a}}q_1\right]_{\mathfrak{g}} = \frac{1}{\sqrt{a}}\alpha^{-1}h = \frac{1}{\sqrt{a}}\sqrt{a}h = \breve{h}.$$

Hence, the Lie algebra generated by $\left\langle \breve{p}_1, \breve{h}, \breve{q}_1, \breve{r}_1 \right\rangle$ is isomorphic to $\mathfrak{he}_1 \times \mathbb{R}$. Concerning evaluations of the commutator $\breve{\mathrm{ad}}$ involving $\breve{z}$, we verify that

$$\begin{aligned} [\breve{z}, \breve{u}]_{\breve{\mathfrak{g}}} &= \alpha^{-1} \left[\alpha\breve{z}, \alpha\breve{u}\right]_{\mathfrak{g}} = \alpha^{-1} \left[\frac{c^2 - bd}{2a^{3/2}b}h - \frac{c}{\sqrt{ab}}r_1 + \frac{1}{\sqrt{a}}z, \alpha\breve{u}\right]_{\mathfrak{g}} = \alpha^{-1} \left[\frac{1}{\sqrt{a}}z, \alpha\breve{u}\right]_{\mathfrak{g}} \\ &= \frac{1}{\sqrt{a}}\alpha^{-1}\delta\alpha\breve{u} = \breve{\delta}\breve{u} \end{aligned}$$

for all $\breve{u} \in \left\langle \breve{p}_1, \breve{h}, \breve{q}_1, \breve{r}_1 \right\rangle$. In matrix notation, the last equality $\frac{1}{\sqrt{a}}\alpha^{-1}\delta\alpha = \breve{\delta}$ requires

$$\begin{pmatrix} \frac{1}{\sqrt{a}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{b}}{\sqrt{a}} \end{pmatrix} . \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & x & 0 & g \\ 1 & 0 & x & 0 \\ 0 & 0 & 1 & p \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{a}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{b}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{s}{a} & 0 \\ 0 & \frac{x}{\sqrt{a}} & 0 & \frac{g}{\sqrt{b}} \\ 1 & 0 & \frac{x}{\sqrt{a}} & 0 \\ 0 & 0 & \frac{\sqrt{b}}{a} & \frac{p}{\sqrt{a}} \end{pmatrix},$$

which we confirm in a straightforward computation. $\square$

Lemma 3.11. $\boxed{\dim\mathfrak{h}=2}$ Any homogeneous triple $(\mathfrak{g},\mathfrak{h},B)$ encountered in Lemma 3.9 is isomorphic to $(\breve{\mathfrak{g}},\breve{\mathfrak{h}},L_4)$ with $\breve{\mathfrak{g}}=\mathfrak{he}_2\rtimes_{\breve{\delta}}\mathbb{R}$, $\breve{\mathfrak{h}}=\langle p_1,p_2\rangle$, and $\breve{\delta}:\mathfrak{he}_2\to\mathfrak{he}_2$ is a derivation.

Proof. The technique of the preceding proof applies here as well. We only state the key terms. According to the table of derivations in Lemma 3.9, we assume δ with respect to $\mathfrak{he}_2=\langle p_1,p_2,h,q_1,q_2\rangle$ is of the form

$$\delta=\begin{pmatrix}0&n&0&t_1&t_2\\0&0&0&t_2&t_3\\0&0&x&0&0\\1&0&0&x&0\\0&1&0&-n&x\end{pmatrix},\qquad x,n,t_i\in\mathbb{R}.\qquad \text{Then,}\quad \breve{\delta}=\begin{pmatrix}0&\frac{n}{\sqrt{a}}&0&\frac{t_1}{a}&\frac{t_2}{a}\\0&0&0&\frac{t_2}{a}&\frac{t_3}{a}\\0&0&\frac{x}{\sqrt{a}}&0&0\\1&0&0&\frac{x}{\sqrt{a}}&0\\0&1&0&-\frac{n}{\sqrt{a}}&\frac{x}{\sqrt{a}}\end{pmatrix},$$

where $a>0$ is the coefficient of the Lorentzian scalar product B in (3.6). We choose the Lie algebra isomorphism $\alpha:\breve{\mathfrak{g}}\to\mathfrak{g}$ as $\alpha=\left(\begin{array}{c|c}I_2&0\\\hline 0&\alpha_{\mathfrak{m}}\end{array}\right)$, where $\alpha_{\mathfrak{m}}$ is adapted from Example 1.1 as

$$\alpha_{\mathfrak{m}}=\begin{pmatrix}\frac{1}{\sqrt{a}}&0&0&-\frac{b}{2a^{3/2}}\\0&\frac{1}{\sqrt{a}}&0&0\\0&0&\frac{1}{\sqrt{a}}&0\\0&0&0&\frac{1}{\sqrt{a}}\end{pmatrix}\quad\text{with inverse}\quad \alpha_{\mathfrak{m}}^{-1}=\begin{pmatrix}\sqrt{a}&0&0&\frac{b}{2\sqrt{a}}\\0&\sqrt{a}&0&0\\0&0&\sqrt{a}&0\\0&0&0&\sqrt{a}\end{pmatrix}.$$

□

Let us summarize the efforts of this chapter in a plain

Corollary 3.12. Suppose the classification in [Ko01] is correct. Then, any 4-dimensional Lorentzian homogeneous triple with isotropy $\dim\mathfrak{h}\geq 1$, and curvature $R_{\mathrm{o}}\neq 0,\mathrm{Ric}_{\mathrm{o}}=0$ is isomorphic to

$$(\mathfrak{g},\mathfrak{h},L_4)\quad\text{with}\quad \mathfrak{g}=\mathfrak{he}_n\times\mathbb{R}^m\rtimes_\delta\mathbb{R},\quad\text{and}\quad \mathfrak{h}=\langle p_1,\dots,p_n\rangle$$

for either $n=1,m=1$, or $n=2,m=0$.

Proof. The corollary summarizes Lemma 3.10, and 3.11. □

The observation motivates us to investigate these types of homogeneous triples for arbitrary integer values n,m in the next chapter. In particular, for $n=0$ we obtain Lorentzian Ricci-flat Lie groups that are not Riemannian-flat.

Chapter 4

Lorentzian Ricci-flat homogeneous triples

We introduce a new class of Lorentzian homogeneous triples $\mathcal{H}^n_m$. To the best of our knowledge, $\mathcal{H}^n_m$ includes all Lorentzian homogeneous triples with geometry $R_\mathrm{o} \neq 0$, $\mathrm{Ric}_\mathrm{o} = 0$ that are mentioned in the literature. In addition, the class covers all homogeneous triples of this particular geometry we could detect.

A homogeneous triple in $\mathcal{H}^n_m$ is denoted by $\mathcal{H}^n_m(Y, \delta)$, which we define in 4.1. In the following section, we clarify what was previously known to us. Then, we motivate the design of the triples in $\mathcal{H}^n_m$. In Section 4.2, we derive the Levi-Civita connection, the Riemannian curvature, and the Ricci tensor of a triple $\mathcal{H}^n_m(Y, \delta)$. The geometry depends on the parameters Y, δ. The curvature $R_\mathrm{o} = 0$, as well as $\mathrm{Ric}_\mathrm{o} = 0$ reduce to non-linear equations in the coefficients of Y, δ. Finally, we state a (complete) list of automorphisms, which map $\mathcal{H}^n_m(Y, \delta)$ to $\mathcal{H}^n_m(\breve{Y}, \breve{\delta})$. The mappings are relevant to classify the triples in $\mathcal{H}^n_m$.

We agree on the following conventions. The basis of the $(2+2n+m)$-dimensional Lie algebra $\mathfrak{g}$ is $\langle p_1, \ldots, p_n, h, q_1, \ldots, q_n, r_1, \ldots, r_m, z\rangle$. We partition $\mathfrak{g}$ into the vector subspaces

$$\mathbf{p} = \langle p_1, \ldots, p_n \rangle\,, \qquad \mathbf{h} = \langle h \rangle\,, \qquad \mathbf{q} = \langle q_1, \ldots, q_n \rangle\,, \qquad \mathbf{r} = \langle r_1, \ldots, r_m \rangle\,, \qquad \mathbf{z} = \langle z \rangle\,.$$

The Lie subalgebra is $\mathfrak{h} = \mathbf{p}$. The Lie algebra decomposes as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where

$$\mathfrak{m} = \langle h, q_1, \ldots, q_n, r_1, \ldots, r_m, z \rangle\,, \qquad \text{equivalently} \qquad \mathfrak{m} = \mathbf{h} \oplus \mathbf{q} \oplus \mathbf{r} \oplus \mathbf{z}.$$

We define $\mathfrak{n} = \mathbf{p} \oplus \mathbf{h} \oplus \mathbf{q} \oplus \mathbf{r}$, and $\mathfrak{m}\backslash z = \mathbf{h} \oplus \mathbf{q} \oplus \mathbf{r}$.

For $u \in \mathfrak{g}$, the vector u_V is the projection on the vector subspace $V \subset \mathfrak{g}$. For instance, $u_\mathbf{r} \in \mathfrak{g}$ is the projection of $u \in \mathfrak{g}$ on $\langle r_1, \ldots, r_m\rangle$. Alternatively, u_V is a vector of V. For instance, $u_{\mathfrak{m}\backslash z}$ is the column vector

$$u_{\mathfrak{m}\backslash z} = \begin{pmatrix} u_\mathbf{h} \\ u_\mathbf{q} \\ u_\mathbf{r} \end{pmatrix}.$$

We state linear mappings with respect to the partition $\mathbf{p}, \mathbf{h}, \mathbf{q}, \mathbf{r}, \mathbf{z}$. For instance, the linear map $\phi : \mathfrak{m}\backslash z \to \mathbf{p} \oplus \mathbf{h}$ with

$$u \mapsto \begin{pmatrix} b & C & 0 \\ x & f^T & g^T \end{pmatrix} .u \;=\; \begin{pmatrix} b.u_\mathbf{h} + C.u_\mathbf{q} \\ x u_\mathbf{h} + f^T.u_\mathbf{q} + g^T.u_\mathbf{r} \end{pmatrix} \qquad \text{for } u \in \mathfrak{m}\backslash z,$$

is defined by $x \in \mathbb{R}$, $b, f \in \mathbb{R}^n$, $g \in \mathbb{R}^m$, $C \in \mathbb{R}^{n\times n}$. 0 denotes an $(n \times m)$-block of zeros.

Let $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a linear mapping. Then, $\alpha_\mathfrak{n} : \mathfrak{n} \to \mathfrak{n}$ denotes the restriction of α to vectors in $\mathfrak{n}$ with $\alpha_\mathfrak{n}.u = \alpha(u)$ for all $u \in \mathfrak{n}$. Analogous, $\alpha_\mathfrak{m} : \mathfrak{m} \to \mathfrak{m}$.

Definition 4.1. $\boxed{\mathcal{H}^n_m(Y,\delta)}$ For integers $n, m \geq 0$, let $\mathfrak{n}$ be the $(1+2n+m)$-dimensional Lie algebra with basis $\langle p_1, \ldots, p_n, h, q_1, \ldots, q_n, r_1, \ldots, r_m \rangle$ and commutator determined by

$$[u_\mathfrak{n}, v_\mathfrak{n}] = \begin{pmatrix} v_\mathbf{p} \\ v_\mathbf{h} \\ v_\mathbf{q} \\ v_\mathbf{r} \end{pmatrix}^T \cdot \begin{pmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & 0 \\ I_n & 0 & Z & 0 \\ 0 & 0 & 0 & Y \end{pmatrix} \cdot \begin{pmatrix} u_\mathbf{p} \\ u_\mathbf{h} \\ u_\mathbf{q} \\ u_\mathbf{r} \end{pmatrix} h \quad \text{for all } u, v \in \mathfrak{n}, \tag{4.1}$$

where $Y \in \mathbb{R}^{m \times m}$, and $Z \in \mathbb{R}^{n \times n}$ are skew symmetric matrices. Then, $\mathcal{H}^n_m(Y,\delta)$ denotes the Lorentzian homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ with Lie algebra $\mathfrak{g} = \mathfrak{n} \rtimes_\delta \mathbb{R}$, and isotropy $\mathfrak{h} = \langle p_1, \ldots, p_n \rangle$. The derivation $\delta : \mathfrak{n} \to \mathfrak{n}$ is

$$u_\mathfrak{n} \mapsto \begin{pmatrix} 0 & 0 & S - xZ/2 & -N^T.Y \\ 0 & x & 0 & g^T \\ I_n & 0 & xI_n + Z & 0 \\ 0 & 0 & N & P \end{pmatrix} \cdot \begin{pmatrix} u_\mathbf{p} \\ u_\mathbf{h} \\ u_\mathbf{q} \\ u_\mathbf{r} \end{pmatrix}, \text{ where } \begin{cases} x \in \mathbb{R},\ S \in \mathbb{R}^{n \times n} \text{ symmetric,} \\ g \in \mathbb{R}^m, N \in \mathbb{R}^{m \times n}, P \in \mathbb{R}^{m \times m}, \\ \text{and } xY = P^T.Y + Y.P. \end{cases} \tag{4.2}$$

The basis element on $\mathbb{R}$ is z such that $[z, u] = \delta.u_\mathfrak{n}$ for all $u \in \mathfrak{g}$. The scalar product on $\mathfrak{m}$ is $B = L_{2+n+m}$ with respect to the basis $\mathfrak{m} = \langle h, q_1, \ldots, q_n, r_1, \ldots, r_m, z \rangle$.

If we omit Y as in $\mathcal{H}^n_m(\delta)$, we assume $Y = 0$. ◇

4.1 Motivation

We are interested in homogeneous triples $(\mathfrak{g}, \mathfrak{h}, B)$ with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ that have special geometric properties:

$$\left.\begin{array}{cl} B \text{ has index } 1 & \text{the scalar product } B \text{ on } \mathfrak{m} \text{ is Lorentzian} \\ R_\mathrm{o} \neq 0 & \text{not Riemannian-flat} \\ \mathrm{Ric}_\mathrm{o} = 0 & \text{Ricci-flat} \end{array}\right\} \tag{4.3}$$

To the best of our knowledge, Definition 4.1 covers all homogeneous triples with geometry (4.3) that are mentioned in the literature:

- [FM05] discuss the geometry of homogeneous triples of the form $\mathcal{H}^n_0(\delta)$. For fixed n, the set of all such triples is a vector space.
- Any symmetric triple with geometry (4.3) is isomorphic to $\mathcal{H}^n_0(\delta)$ for a suitable derivation δ, [CW70]. We supplement Corollary 4.12 with details.

In addition, the construction covers all homogeneous triples of this particular geometry we could detect:

- The geometry of triples $\mathcal{H}^n_m(Y,\delta)$ with $m \neq 0$ has not been published yet. Due to the non-linear constraint $xY = P^T.Y + Y.P$, the set of all triples of the form $\mathcal{H}^n_m(Y,\delta)$ for fixed n, and $m \geq 2$ is not a vector space.

- For $n = 0$, we have isotropy of $\dim \mathfrak{h} = 0$. Then, the corresponding homogeneous space is a Lie group with left-invariant metric.

- Suppose Komrakov's list is a complete classification of 4-dimensional homogeneous pairs. Then, any 4-dimensional homogeneous triple with isotropy $\dim \mathfrak{h} \geq 1$ and geometry (4.3) is isomorphic to $\mathcal{H}^1_1(\delta)$, or $\mathcal{H}^2_0(\delta)$ for a suitable derivation. Corollary 4.3 gives the proof.

To get familiar with the construction, we encourage the reader to have a quick glance at discussions 5.1, 5.2, 5.4. There, we give an algebraic, as well as geometric description of the 4-dimensional homogeneous triples that originate from Definition 4.1.

In this section, we illuminate the design of the derivation $\delta : \mathfrak{n} \to \mathfrak{n}$ in (4.2). The aspects we investigate are the ρ-invariance of the scalar product $B = L_{2+n+m}$, the Jacobi identity of the commutator, and finding a reductive decomposition.

ρ-invariance of the scalar product

Lets consider a homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ that coincides with $\mathcal{H}^n_m(Y, \delta)$, except the derivation $\delta : \mathfrak{n} \to \mathfrak{n}$ is *any* derivation on $\mathfrak{n}$. We partition the action of the linear mapping δ restricted to elements of $\mathfrak{h}$ as

$$\delta(u_{\mathbf{p}}) = \begin{pmatrix} A \\ e^T \\ H \\ L \end{pmatrix} .u_{\mathbf{p}} \quad \text{for all } u \in \mathfrak{h}, \text{ where} \quad A \in \mathbb{R}^{n\times n}, e \in \mathbb{R}^n, H \in \mathbb{R}^{n\times n}, L \in \mathbb{R}^{m\times n}.$$

Then, the isotropy representation $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m})$ of $\mathfrak{h}$ on $\mathfrak{m}$ is determined by the commutator relations

$$\left.\begin{aligned} [u_{\mathbf{p}}, h] &= 0 \\ [u_{\mathbf{p}}, v_{\mathbf{q}}] &= (u_{\mathbf{p}}^T.v_{\mathbf{q}})h \\ [u_{\mathbf{p}}, v_{\mathbf{r}}] &= 0 \\ [u_{\mathbf{p}}, z] &= -\delta(u_{\mathbf{p}}) \end{aligned}\right\} \quad \text{as} \quad \rho(u_{\mathbf{p}}) = \begin{pmatrix} 0 & u_{\mathbf{p}}^T & 0 & -e^T.u_{\mathbf{p}} \\ 0 & 0 & 0 & -H.u_{\mathbf{p}} \\ 0 & 0 & 0 & -L.u_{\mathbf{p}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{for all } u_{\mathbf{p}} \in \mathfrak{h}.$$

We express the ρ-invariance of the $(0,2)$-tensor B as $\rho(u_{\mathbf{p}})^T.B + B.\rho(u_{\mathbf{p}}) = 0$. Since $B = L_{2+n+m}$, the two matrix products

$$\rho(u_{\mathbf{p}})^T.B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{\mathbf{p}} \\ 0 & 0 & 0 & 0 \\ 0 & -u_{\mathbf{p}}^T.H^T & -u_{\mathbf{p}}^T.L^T & -u_{\mathbf{p}}^T.e \end{pmatrix}, \quad \text{and} \quad B.\rho(u_{\mathbf{p}}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -H.u_{\mathbf{p}} \\ 0 & 0 & 0 & -L.u_{\mathbf{p}} \\ 0 & u_{\mathbf{p}}^T & 0 & -e^T.u_{\mathbf{p}} \end{pmatrix}$$

reduce the ρ-invariance of B to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (I_n - H).u_{\mathbf{p}} \\ 0 & 0 & 0 & -L.u_{\mathbf{p}} \\ 0 & u_{\mathbf{p}}^T.(I_n - H^T) & -u_{\mathbf{p}}^T.L^T & -2e^T.u_{\mathbf{p}} \end{pmatrix} = 0 \quad \text{for all } u_{\mathbf{p}} \in \mathbb{R}^n.$$

The equation imposes $\boxed{e, L = 0, \text{ and } H = I_n}$ on the derivation $\delta : \mathfrak{n} \to \mathfrak{n}$.

Jacobi identity

Lets consider a homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ that coincides with $\mathcal{H}_m^n(Y, \delta)$, where $Z = 0$, and the linear mapping $\delta : \mathfrak{n} \to \mathfrak{n}$ is given by

$$u_{\mathfrak{n}} \mapsto \begin{pmatrix} A & b & C & D \\ 0 & x & f^T & g^T \\ I_n & i & J & K \\ 0 & l & N & P \end{pmatrix} . \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix} \qquad \text{for all } u \in \mathfrak{n},$$

where $A, b, C, \ldots, P$ denote matrices and vectors of appropriate format with a priori arbitrary entries. However, from the previous discussion we know that $B = L_{2+n+m}$ is ρ-invariant.

We want to assure that the commutator of the Lie algebra $\mathfrak{g}$ satisfies the Jacobi identity. Equivalently, we find restrictions on $A, b, C, \ldots, P$, that make $\delta : \mathfrak{n} \to \mathfrak{n}$ a derivation. Recall from the introduction, $\delta : \mathfrak{n} \to \mathfrak{n}$ is a derivation if the linear mapping

$$\xi : \mathfrak{n} \to \mathfrak{gl}(\mathfrak{n}) \quad \text{with} \quad \xi(v).u = \delta.[u, v] - [\delta.u, v] - [u, \delta.v]$$

is the zero mapping for all $v \in \mathfrak{n}$. In Table 4.2, we derive $\xi(v)$ separately for values $v_{\mathbf{p}}, v_{\mathbf{h}}, v_{\mathbf{q}}, v_{\mathbf{r}} \in \mathfrak{n}$. Due to linearity, it suffices to have $\xi(v_{\mathbf{p}}) = 0, \xi(v_{\mathbf{h}}) = 0, \ldots$.

- $\xi(v_{\mathbf{p}})$ is the sum of the three matrices below.

$\delta.[\circ, v_{\mathbf{p}}]$	$-[\delta(\circ), v_{\mathbf{p}}]$	$-[\circ, \delta.v_{\mathbf{p}}]$
$\begin{pmatrix} 0 & 0 & -b.v_{\mathbf{p}}^T & 0 \\ 0 & 0 & -x.v_{\mathbf{p}}^T & 0 \\ 0 & 0 & -i.v_{\mathbf{p}}^T & 0 \\ 0 & 0 & -l.v_{\mathbf{p}}^T & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ v_{\mathbf{p}}^T.I_n & v_{\mathbf{p}}^T.i & v_{\mathbf{p}}^T.J & v_{\mathbf{p}}^T.K \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -v_{\mathbf{p}}^T.I_n & 0 & v_{\mathbf{p}}^T.A^T & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\xi(v_{\mathbf{p}}) = 0$ for all $v \in \mathfrak{n}$ requires $\boxed{b, i, l, K = 0, \text{ and } J = xI_n - A^T}$. Henceforth, we assume $\delta : \mathfrak{n} \to \mathfrak{n}$ is of the form

$$u_{\mathfrak{n}} \mapsto \begin{pmatrix} A & 0 & C & D \\ 0 & x & f^T & g^T \\ I_n & 0 & J & 0 \\ 0 & 0 & N & P \end{pmatrix} . \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix} \qquad \text{for all } u \in \mathfrak{n}.$$

For a better overview, the matrix J has not been substituted by $xI_n - A^T$.

- $\xi(v_{\mathbf{h}}) = 0$, since $\delta.[u, v_{\mathbf{h}}] = \delta(0) = 0$, $-[\delta.u, v_{\mathbf{h}}] = 0$, and $-[u, \delta(v_{\mathbf{h}})] = -[u, xv_{\mathbf{h}}h] = 0$ for all $u \in \mathfrak{n}$.
- $\xi(v_{\mathbf{q}})$ is the sum of the three matrices below.

$\delta.[\circ, v_{\mathbf{q}}]$	$-[\delta(\circ), v_{\mathbf{q}}]$	$-[\circ, \delta.v_{\mathbf{q}}]$
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ xv_{\mathbf{q}}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -v_{\mathbf{q}}^T.A & 0 & -v_{\mathbf{q}}^T.C & -v_{\mathbf{q}}^T.D \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -v_{\mathbf{q}}^T.J^T & 0 & v_{\mathbf{q}}^T.C^T & -v_{\mathbf{q}}^T.N^T.Y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\xi(v_{\mathbf{q}}) = 0$ for all $v \in \mathfrak{n}$ results in the additional relations $\boxed{C^T = C, \text{ and } D = -N^T.Y}$.

- $\xi(v_{\mathbf{r}})$ is the sum of the three matrices below.

$\delta.[\circ, v_{\mathbf{r}}]$	$-[\delta(\circ), v_{\mathbf{r}}]$	$-[\circ, \delta.v_{\mathbf{r}}]$
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_{\mathbf{r}}^T.xY \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -v_{\mathbf{r}}^T.Y.N & -v_{\mathbf{r}}^T.Y.P \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & v_{\mathbf{r}}^T.D^T & -v_{\mathbf{r}}^T.P^T.Y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\xi(v_{\mathbf{r}}) = 0$ for all $v \in \mathfrak{n}$ requires $\boxed{xY = P^T.Y + Y.P}$.

Reductive decomposition

So far, the discussion has shown that on the Lie algebra $\mathfrak{n}$ with commutator (4.1), where $Z = 0$, a derivation $\delta : \mathfrak{n} \to \mathfrak{n}$ is of the form

$$u_{\mathfrak{n}} \mapsto \begin{pmatrix} A & & C & -N^T.Y \\ & x & f^T & g^T \\ I_n & & xI_n - A^T & \\ & & N & P \end{pmatrix} . \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix}, \text{ where } \begin{cases} x \in \mathbb{R},\ A, C \in \mathbb{R}^{n\times n}, \\ C \text{ symmetric}, \\ g \in \mathbb{R}^m, N \in \mathbb{R}^{m\times n}, P \in \mathbb{R}^{m\times m}, \\ \text{and } xY = P^T.Y + Y.P \end{cases} \tag{4.4}$$

with 0's omitted. We define $\mathrm{H}^n_m(Y, \delta)$ to be the homogeneous triple $(\mathfrak{n} \rtimes_\delta \mathbb{R}, \mathbf{p}, L_{2+n+m})$. The notion of $\mathrm{H}^n_m(Y, \delta)$ is used only to state the two following results.

If $A \neq 0$ the triple $\mathrm{H}^n_m(Y, \delta)$ is not in a reductive decomposition. We have $[\mathfrak{h}, \mathfrak{m}] \not\subset \mathfrak{m}$, since

$$[u_{\mathbf{p}}, z]_{\mathfrak{h}} = -[z, u_{\mathbf{p}}]_{\mathfrak{h}} = -\delta(u_{\mathbf{p}})_{\mathfrak{h}} = -A.u_{\mathbf{p}} \in \mathfrak{h}.$$

However, the next result proves that $\mathrm{H}^n_m(Y, \delta)$ is isomorphic to $\mathcal{H}^n_m(Y, \breve{\delta})$, where $\breve{\delta}$ is an appropriate derivation that is in the scope of Definition 4.1. It turns out that the symmetric part of the matrix A as well as the vector f are redundant by isomorphy.

Lemma 4.2. $\mathrm{H}^n_m(Y,\delta)$ is isomorphic to $\mathcal{H}^n_m(Y,\breve{\delta})$ where $\breve{\delta}:\mathfrak{n}\to\mathfrak{n}$ is defined by

$$u_{\mathfrak{n}} \mapsto \begin{pmatrix} & S - xZ/2 & -N^T.Y \\ x & & g^T \\ I_n & xI_n + Z & \\ & N & P \end{pmatrix} . \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix}, \text{ with } \begin{cases} Z = A - A^T, \text{ and} \\ S = C + A.A^T - x(A + A^T)/2. \end{cases}$$

Proof. We denote the commutator of $\mathrm{H}^n_m(Y,\delta)$ by ad, and the commutator of $\mathcal{H}^n_m(Y,\breve{\delta})$ by $\breve{\mathrm{ad}}$. In the spirit of Remark 1.9, we show that the diagram

$$\begin{array}{rccccc} \breve{\mathrm{ad}}: & \mathfrak{g} & \times & \mathfrak{g} & \to & \mathfrak{g} \\ & \downarrow \alpha & & \downarrow \alpha & & \uparrow \alpha^{-1} \\ \mathrm{ad}: & \mathfrak{g} & \times & \mathfrak{g} & \to & \mathfrak{g} \end{array}$$

commutes for

$$\alpha = \begin{pmatrix} I_n & 0 & A & 0 & -f \\ & 1 & 0 & 0 & 0 \\ & & I_n & 0 & 0 \\ & & & I_m & 0 \\ & & & & 1 \end{pmatrix}, \quad \text{and} \quad \alpha^{-1} = \begin{pmatrix} I_n & 0 & -A & 0 & f \\ & 1 & 0 & 0 & 0 \\ & & I_n & 0 & 0 \\ & & & I_m & 0 \\ & & & & 1 \end{pmatrix}.$$

Equivalently, we prove $[u,v]_{\breve{\mathrm{ad}}} = \alpha^{-1}[\alpha.u,\alpha.v]$ for all $u,v\in\mathfrak{g}$, where the commutator $[\cdot,\cdot]$ without subindex denotes the action of ad. By linearity, the equation decomposes into

$$[u_{\mathfrak{n}}, v_{\mathfrak{n}}]_{\breve{\mathrm{ad}}} = \alpha^{-1}[\alpha_{\mathfrak{n}}.u_{\mathfrak{n}}, \alpha_{\mathfrak{n}}.v_{\mathfrak{n}}] \tag{4.5}$$

$$[z, u_{\mathfrak{n}}]_{\breve{\mathrm{ad}}} = \alpha^{-1}[\alpha(z), \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}] \tag{4.6}$$

for all $u,v\in\mathfrak{g}$.

The commutator in (4.5) reduces to matrix multiplication. We confirm

$$\begin{aligned} &\alpha^{-1}[\alpha_{\mathfrak{n}}.u_{\mathfrak{n}}, \alpha_{\mathfrak{n}}.v_{\mathfrak{n}}] \\ &= \big(v_{\mathfrak{n}}^T.\alpha_{\mathfrak{n}}^T. \begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & 0 & \\ & & & Y \end{pmatrix} . \begin{pmatrix} I_n & & A & \\ & 1 & & \\ & & I_n & \\ & & & I_m \end{pmatrix} .u_{\mathfrak{n}}\big)\alpha(h) \\ &= \big(v_{\mathfrak{n}}^T. \begin{pmatrix} I_n & & & \\ & 1 & & \\ A^T & & I_n & \\ & & & I_m \end{pmatrix} . \begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & A & \\ & & & Y \end{pmatrix} .u_{\mathfrak{n}}\big)h = \big(v_{\mathfrak{n}}^T. \begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & A - A^T & \\ & & & Y \end{pmatrix} .u_{\mathfrak{n}}\big)h. \end{aligned}$$

The commutator in (4.6) is slightly more complicated. We have

$$\alpha^{-1}\left[\alpha z, \alpha u_{\mathfrak{n}}\right] = \alpha^{-1}\left[\begin{pmatrix} -f \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}\right] = \alpha^{-1}\left[z, \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}\right] + \alpha^{-1}\left[\begin{pmatrix} -f \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}\right].$$

The first summand is

$$\begin{aligned}
&\alpha^{-1}\left[z, \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}\right] = \alpha_{\mathfrak{n}}^{-1}.\delta.\alpha_{\mathfrak{n}}.u_{\mathfrak{n}} \\
&= \alpha_{\mathfrak{n}}^{-1}.\begin{pmatrix} A & & C & -N^T.Y \\ & x & f^T & g^T \\ I_n & & xI_n - A^T & \\ & & N & P \end{pmatrix}.\begin{pmatrix} I_n & & A & \\ & 1 & & \\ & & I_n & \\ & & & I_m \end{pmatrix}.u_{\mathfrak{n}} \\
&= \begin{pmatrix} I_n & & -A & \\ & 1 & & \\ & & I_n & \\ & & & I_m \end{pmatrix}.\begin{pmatrix} A & & C + A.A & -N^T.Y \\ & x & f^T & g^T \\ I_n & & xI_n + A - A^T & \\ & & N & P \end{pmatrix}.u_{\mathfrak{n}} \\
&= \begin{pmatrix} & & C + A.A^T - xA & -N^T.Y \\ & x & f^T & g^T \\ I_n & & xI_n + A - A^T & \\ & & N & P \end{pmatrix}.u_{\mathfrak{n}} = \begin{pmatrix} & & S - xZ/2 & -N^T.Y \\ & x & f^T & g^T \\ I_n & & xI_n + Z & \\ & & N & P \end{pmatrix}.u_{\mathfrak{n}}.
\end{aligned}$$

The second summand simplifies to

$$\begin{aligned}
\ldots = \alpha^{-1}\left[\alpha_{\mathfrak{n}}.u_{\mathfrak{n}}, \begin{pmatrix} -f \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right] &= \Big(\begin{pmatrix} -f \\ 0 \\ 0 \\ 0 \end{pmatrix}^T.\begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & Y \end{pmatrix}.\alpha_{\mathfrak{n}}.u_{\mathfrak{n}}\Big)h \\
&= \Big(\begin{pmatrix} 0 \\ 0 \\ -f \\ 0 \end{pmatrix}^T.\begin{pmatrix} I_n & & A & \\ & 1 & & \\ & & I_n & \\ & & & I_m \end{pmatrix}.u_{\mathfrak{n}}\Big)h = \begin{pmatrix} & & 0 & 0 \\ & 0 & -f^T & 0 \\ 0 & & 0 & \\ & & 0 & 0 \end{pmatrix}.u_{\mathfrak{n}}.
\end{aligned}$$

The sum of both matrix expressions confirms $[z, u_{\mathfrak{n}}]_{\breve{\mathrm{ad}}} = \alpha^{-1}[\alpha(z), \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}] = \breve{\delta}.u_{\mathfrak{n}}$ for all $u \in \mathfrak{g}$. □

The next result is a mere application of the previous lemma.

Corollary 4.3. Suppose the classification in [Ko01] is correct. Then, any 4-dimensional Lorentzian homogeneous triple with isotropy $\dim \mathfrak{h} \geq 1$, and curvature $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$ is isomorphic to $\mathcal{H}_1^1(\delta)$, or $\mathcal{H}_0^2(\delta)$.

Proof. According to Lemma 3.10, any such homogeneous triple with 1-dimensional isotropy is isomorphic to $(\mathfrak{g}_1, \mathfrak{h}_1, L_4)$ with $\mathfrak{g}_1 = \mathfrak{he}_1 \times \mathbb{R} \rtimes_{\delta_1} \mathbb{R}$, $\mathfrak{h}_1 = \langle p_1 \rangle$. According to Lemma 3.11, any such triple with 2-dimensional isotropy is isomorphic to $(\mathfrak{g}_2, \mathfrak{h}_2, L_4)$ with $\mathfrak{g}_2 = \mathfrak{he}_2 \rtimes_{\delta_2} \mathbb{R}$, $\mathfrak{h}_2 = \langle p_1, p_2 \rangle$. The lemmas state the derivations as

$$\delta_1 = \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & x & 0 & g \\ 1 & 0 & x & 0 \\ 0 & 0 & n & p \end{pmatrix}, \quad \text{and} \quad \delta_2 = \begin{pmatrix} 0 & n & 0 & t_1 & t_2 \\ 0 & 0 & 0 & t_2 & t_3 \\ 0 & 0 & x & 0 & 0 \\ 1 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & -n & x \end{pmatrix} \quad \text{with coefficients in } \mathbb{R}.$$

The derivation δ_1 is in the scope of Definition 4.1. The triple $(\mathfrak{g}_1, \mathfrak{h}_1, L_4)$ coincides with $\mathcal{H}^1_1(\delta_1)$, whereas $(\mathfrak{g}_2, \mathfrak{h}_2, L_4)$ coincides with $\mathrm{H}^2_0(\delta_2)$. However, according to the previous lemma, $\mathrm{H}^2_0(\delta_2)$ is isomorphic to $\mathcal{H}^2_0(\breve{\delta}_2)$ for an appropriate derivation $\breve{\delta}_2$.

As a result of Section 3.1, no effective 4-dimensional Lorentzian homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ with isotropy $\dim \mathfrak{h} > 2$, and geometry $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$ exist. $\square$

4.2 Geometry

In this section, we derive the geometry of the homogeneous triple $\mathcal{H}^n_m(Y, \delta)$. The associated geometric tensors $\nu, \Lambda, R_{\mathrm{o}}, \mathrm{Ric}_{\mathrm{o}}$ depend on the coefficients of Y, δ.

Lemma 4.4. The homogeneous triple $\mathcal{H}^n_m(Y, \delta)$ determines the tensor $\nu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ as

$$\nu(u_{\mathfrak{n}}, v_{\mathfrak{n}}) = \begin{pmatrix} v_{\mathbf{p}} \\ v_{\mathbf{h}} \\ v_{\mathbf{q}} \\ v_{\mathbf{r}} \end{pmatrix}^T \cdot \begin{pmatrix} 0 & 0 & I_n/2 & 0 \\ 0 & 0 & 0 & 0 \\ I_n/2 & 0 & xI_n & N^T/2 \\ 0 & 0 & N/2 & (P+P^T)/2 \end{pmatrix} \cdot \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix} h$$

$$\nu(u, z) = \begin{pmatrix} 0 & x/2 & 0 & g^T/2 & 0 \\ -I_n/2 & 0 & -xI_n/2 & -N^T/2 & 0 \\ 0 & 0 & 0 & -(Y+P^T)/2 & -g \\ 0 & 0 & 0 & 0 & -x \end{pmatrix} \cdot \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \\ u_{\mathbf{z}} \end{pmatrix}$$

for all $u, v \in \mathfrak{g}$.

Proof. Recall, ν is defined implicitly by $2B\big(\nu(u,v), w_{\mathfrak{m}}\big) = B\big(u_{\mathfrak{m}}, [w,v]_{\mathfrak{m}}\big) + B\big([w,u]_{\mathfrak{m}}, v_{\mathfrak{m}}\big)$ for all $u, v, w \in \mathfrak{g}$. We compute $\nu(u_{\mathfrak{n}}, v_{\mathfrak{n}})$ in Table 4.3, while $\nu(u, z)$ follows from Table 4.4. By symmetry and linearity, both evaluations determine $\nu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$. $\square$

Definition 4.5. We define $\mathcal{P} : \mathbb{R}^{n+m} \to \mathfrak{gl}(\mathfrak{m})$ and $\mathcal{Q} : \mathbb{R} \times \mathbb{R}^{n+m} \times \mathbb{R}^{(n+m)\times(n+m)} \to \mathfrak{gl}(\mathfrak{m})$ as

$$\mathcal{P}(f) = \begin{pmatrix} 0 & f^T & 0 \\ 0 & 0 & -f \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{Q}(x, g, X) = \begin{pmatrix} x & g^T & 0 \\ 0 & X & -g \\ 0 & 0 & -x \end{pmatrix}.$$

Lemma 4.6. The homogeneous triple $\mathcal{H}^n_m(Y,\delta)$ determines the Levi-Civita connection $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ as

$$\Lambda(u_\mathfrak{n}) = \mathcal{P}\left(\begin{pmatrix} I_n & 0 & xI_n + Z/2 & N^T/2 \\ 0 & 0 & N/2 & (P+P^T+Y)/2 \end{pmatrix} . \begin{pmatrix} u_\mathbf{p} \\ u_\mathbf{h} \\ u_\mathbf{q} \\ u_\mathbf{r} \end{pmatrix}\right)$$

$$\Lambda(z) = \mathcal{Q}\left(x, \begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} Z/2 & -N^T/2 \\ N/2 & (P-Y-P^T)/2 \end{pmatrix}\right) = \begin{pmatrix} x & 0 & g^T & 0 \\ 0 & Z/2 & -N^T/2 & 0 \\ 0 & N/2 & (P-Y-P^T)/2 & -g \\ 0 & 0 & 0 & -x \end{pmatrix}$$

for all $u \in \mathfrak{g}$.

Proof. The formula for the Levi-Civita connection is $\Lambda(u).v_\mathfrak{m} = \frac{1}{2}[u,v]_\mathfrak{m} + \nu(u,v)$ for all $u, v \in \mathfrak{g}$. The previous lemma provides the values for $\nu(u,v)$. We yield Λ in a straightforward computation carried out in Table 4.5. □

Remark 4.7. Let $x \in \mathbb{R}$, $f, g \in \mathbb{R}^{n+m}$, and $X \in \mathbb{R}^{(n+m)\times(n+m)}$ skew symmetric. We have

$$\big[\mathcal{P}(f), \mathcal{P}(g)\big]_{\mathfrak{gl}} = 0 \tag{4.7}$$

$$\big[\mathcal{P}(f), \mathcal{Q}(x,g,X)\big]_{\mathfrak{gl}} = -\mathcal{P}\big((xI_{n+m} + X).f\big). \tag{4.8}$$

$\big[\mathcal{P}(f), \mathcal{P}(g)\big]_{\mathfrak{gl}} = 0$ is a consequence of (1.16). The second commutator computes as

$$\mathcal{P}(f).\mathcal{Q}(x,g,X) = \begin{pmatrix} 0 & f^T.X & -f^T.g \\ 0 & 0 & xf \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{Q}(x,g,X).\mathcal{P}(f) = \begin{pmatrix} 0 & xf^T & -g^T.f \\ 0 & 0 & -X.f \\ 0 & 0 & 0 \end{pmatrix}$$

$$\big[\mathcal{P}(f), \mathcal{Q}(x,g,X)\big]_{\mathfrak{gl}} = \begin{pmatrix} 0 & -f^T.(xI_{n+m} + X^T) & 0 \\ 0 & 0 & (xI_{n+m} + X).f \\ 0 & 0 & 0 \end{pmatrix}$$

◇

Lemma 4.8. The non-zero evaluations of the Riemannian curvature tensor $R_\mathrm{o} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ induced by the homogeneous triple $\mathcal{H}^n_m(Y,\delta)$ are determined by

$$R_\mathrm{o}(u,z) = \frac{1}{4}\mathcal{P}\left(\begin{pmatrix} C_{\mathbf{q},\mathbf{q}} & C_{\mathbf{q},\mathbf{r}} \\ C_{\mathbf{r},\mathbf{q}} & C_{\mathbf{r},\mathbf{r}} \end{pmatrix} . \begin{pmatrix} u_\mathbf{q} \\ u_\mathbf{r} \end{pmatrix}\right) \quad \text{for all } u \in \mathfrak{m}, \tag{4.9}$$

where

$$C_{\mathbf{q},\mathbf{q}} = 4S + 3N^T.N + Z.Z$$

$$C_{\mathbf{r},\mathbf{r}} = 2(P+P^T).(P - xI_m) + (Y + P^T - P).(Y + P + P^T) - 2P^T.Y - N.N^T$$

$$C_{\mathbf{q},\mathbf{r}} = N^T.(3P + P^T - 3Y) - (2xI_n + Z).N^T$$

$$C_{\mathbf{r},\mathbf{q}} = C^T_{\mathbf{q},\mathbf{r}} = (3P^T + P + 3Y).N - N.(2xI_n - Z).$$

Proof. The Riemannian curvature tensor satisfies

$$R_{\mathrm{o}}(u,v) = \big[\Lambda(u),\Lambda(v)\big]_{\mathfrak{gl}} - \Lambda\big([u,v]\big) \qquad \text{for all } u,v\in\mathfrak{m}.$$

• $R_{\mathrm{o}}(u_{\mathfrak{m}\backslash z}, v_{\mathfrak{m}\backslash z})$ vanishes for vectors $u,v\in\mathfrak{m}\backslash z$. According to Lemma 4.6, $\Lambda(u_{\mathfrak{m}\backslash z}) = \mathcal{P}(f_u)$, and $\Lambda(v_{\mathfrak{m}\backslash z}) = \mathcal{P}(f_v)$ for appropriate $f_u, f_v\in\mathbb{R}^{n+m}$. But then $\big[\Lambda(u_{\mathfrak{m}\backslash z}),\Lambda(v_{\mathfrak{m}\backslash z})\big]_{\mathfrak{gl}} = 0$ by the previous remark. Furthermore, $[u_{\mathfrak{m}\backslash z}, v_{\mathfrak{m}\backslash z}]\in\langle h\rangle$ so that $\Lambda\big([u_{\mathfrak{m}\backslash z}, v_{\mathfrak{m}\backslash z}]\big) = 0$.

• Table 4.6 yields $R_{\mathrm{o}}(u_{\mathfrak{m}\backslash z}, z)$ as

$$R_{\mathrm{o}}(u_{\mathfrak{m}\backslash z}, z) = \frac{1}{4}\mathcal{P}\left(\big(-A_1 + A_2\big).\begin{pmatrix} u_{\mathbf{q}} \\ u_{\mathbf{r}}\end{pmatrix}\right) \qquad \text{for all } u\in\mathfrak{m}.$$

Up to the factor 4, the matrix $-A_1$ is the contribution of $\big[\Lambda(u_{\mathfrak{m}\backslash z}),\Lambda(z)\big]_{\mathfrak{gl}}$, and A_2 is the contribution of $-\Lambda\big([u_{\mathfrak{m}\backslash z}, z]\big)$. Table 4.7 simplifies the sum $-A_1 + A_2$ and partitions the matrix as in (4.9).

• Skew symmetry of the Riemannian curvature tensor gives $R_{\mathrm{o}}(z,z) = -R_{\mathrm{o}}(z,z) = 0$. □

Lemma 4.9. The non-zero evaluations of the Ricci tensor $\mathrm{Ric}_{\mathrm{o}} : \mathfrak{m}\times\mathfrak{m}\to\mathbb{R}$ induced by the homogeneous triple $\mathcal{H}^n_m(Y,\delta)$ are determined by

$$\mathrm{Ric}_{\mathrm{o}}(z,z) = -\frac{1}{4}\,\mathrm{tr}\,\big(4S + 3N^T.N + Z.Z\big) - \frac{1}{4}\,\mathrm{tr}\,\big(2(P+P^T).(P - xI_m) + Y.Y - N.N^T\big) \tag{4.10}$$

Proof. The Ricci tensor satisfies $\mathrm{Ric}_{\mathrm{o}}(u,v) = \mathrm{tr}\,(w\mapsto R_{\mathrm{o}}(w,u).v)$ for all $u,v\in\mathfrak{m}$.

First, we obtain $\mathrm{Ric}_{\mathrm{o}}(u_{\mathfrak{m}\backslash z}, v) = 0$ for all $u,v\in\mathfrak{m}$.

$$\begin{aligned}
\mathrm{Ric}_{\mathrm{o}}(u_{\mathfrak{m}\backslash z}, v) &= \mathrm{tr}\,\big(w\mapsto R_{\mathrm{o}}(w, u_{\mathfrak{m}\backslash z}).v\big) = \mathrm{tr}\,\big(w\mapsto R_{\mathrm{o}}(w_{\mathbf{z}}, u_{\mathfrak{m}\backslash z}).v\big) \\
&= \mathrm{tr}\left(w\mapsto -\frac{w_{\mathbf{z}}}{4}\mathcal{P}\left(\begin{pmatrix} C_{\mathbf{q},\mathbf{q}} & C_{\mathbf{q},\mathbf{r}} \\ C_{\mathbf{r},\mathbf{q}} & C_{\mathbf{r},\mathbf{r}}\end{pmatrix}.\begin{pmatrix} u_{\mathbf{q}} \\ u_{\mathbf{r}}\end{pmatrix}\right).v\right) \\
&= -\frac{1}{4}z^T.\mathcal{P}\left(\begin{pmatrix} C_{\mathbf{q},\mathbf{q}} & C_{\mathbf{q},\mathbf{r}} \\ C_{\mathbf{r},\mathbf{q}} & C_{\mathbf{r},\mathbf{r}}\end{pmatrix}.\begin{pmatrix} u_{\mathbf{q}} \\ u_{\mathbf{r}}\end{pmatrix}\right).v = 0
\end{aligned}$$

By symmetry, $\mathrm{Ric}_{\mathrm{o}}(u, v_{\mathfrak{m}\backslash z}) = 0$ for all $u,v\in\mathfrak{m}$.

What remains is the computation of

$$\begin{aligned}
\mathrm{Ric}_{\mathrm{o}}(z,z) &= \mathrm{tr}\,\big(w\mapsto R_{\mathrm{o}}(w,z).z\big) \\
&= \mathrm{tr}\left(w\mapsto \frac{1}{4}\mathcal{P}\left(\begin{pmatrix} C_{\mathbf{q},\mathbf{q}} & C_{\mathbf{q},\mathbf{r}} \\ C_{\mathbf{r},\mathbf{q}} & C_{\mathbf{r},\mathbf{r}}\end{pmatrix}.\begin{pmatrix} w_{\mathbf{q}} \\ w_{\mathbf{r}}\end{pmatrix}\right).z\right) \\
&= \mathrm{tr}\left(w\mapsto -\frac{1}{4}\begin{pmatrix} C_{\mathbf{q},\mathbf{q}} & C_{\mathbf{q},\mathbf{r}} \\ C_{\mathbf{r},\mathbf{q}} & C_{\mathbf{r},\mathbf{r}}\end{pmatrix}.\begin{pmatrix} w_{\mathbf{q}} \\ w_{\mathbf{r}}\end{pmatrix}\right) \\
&= -\frac{1}{4}\,\mathrm{tr}\,C_{\mathbf{q},\mathbf{q}} - \frac{1}{4}\,\mathrm{tr}\,C_{\mathbf{r},\mathbf{r}}
\end{aligned}$$

The terms $\mathrm{tr}\,C_{\mathbf{q},\mathbf{q}}$ and $\mathrm{tr}\,C_{\mathbf{r},\mathbf{r}}$ are derived in Table 4.8 and match those in (4.10). □

Remark 4.10. According to [KN69] p.204, the holonomy algebra of a homogeneous triple is given by

$$\mathfrak{c} + [\Lambda(\mathfrak{g}), \mathfrak{c}]_{\mathfrak{gl}} + [\Lambda(\mathfrak{g}), [\Lambda(\mathfrak{g}), \mathfrak{c}]]_{\mathfrak{gl}} + \ldots \tag{4.11}$$

where $\mathfrak{c} = \{R_{\mathrm{o}}(u, v) : u, v \in \mathfrak{m}\}$ is the set of all curvature endomorphisms.

The holonomy of $\mathcal{H}^n_m(Y, \delta)$ is abelian. To see this, we recall the results on the Levi-Civita connection and the Riemannian curvature. Due to the commutator relations (4.7), (4.8), all summands in (4.11) are subsets of $\{\mathcal{P}(f) : f \in \mathbb{R}^{n+m}\}$. We have $[\mathcal{P}(f), \mathcal{P}(g)]_{\mathfrak{gl}} = 0$ for all $f, g \in \mathbb{R}^{n+m}$. ◇

Special instances

Previously, we have derived the geometry of the homogeneous triple $\mathcal{H}^n_m(Y, \delta)$. Thus, we easily obtain two corollaries that specialize on the geometry of triples of the form $\mathcal{H}^0_m(Y, \delta)$, and $\mathcal{H}^n_0(\delta)$.

A homogeneous triple $\mathcal{H}^0_m(Y, \delta)$ corresponds to a Lie algebra with Lorentzian scalar product. The set of all such triples is not a vector space due to the non-linear relation $xY = P^T.Y + Y.P$.

Corollary 4.11. $\boxed{\mathcal{H}^0_m(Y, \delta)}$ Let $\mathfrak{n}$ be the $(1+m)$-dimensional Lie algebra with basis $\langle h, r_1, \ldots, r_m\rangle$ and commutator determined by

$$[u_{\mathfrak{n}}, v_{\mathfrak{n}}] = \begin{pmatrix} v_{\mathbf{h}} \\ v_{\mathbf{r}} \end{pmatrix}^T . \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} . \begin{pmatrix} u_{\mathbf{h}} \\ u_{\mathbf{r}} \end{pmatrix} h \quad \text{for all } u, v \in \mathfrak{n},$$

where $Y \in \mathbb{R}^{m\times m}$ is a skew symmetric matrix. The homogeneous triple $\big(\mathfrak{g}, \{0\}, B\big)$ with Lie algebra $\mathfrak{g} = \mathfrak{n} \rtimes_\delta \mathbb{R}$, isotropy $\{0\}$, scalar product $B = L_{2+m}$ with respect to $\langle h, r_1, \ldots, r_m, z\rangle$, and derivation $\delta : \mathfrak{n} \to \mathfrak{n}$ as

$$u_{\mathfrak{n}} \mapsto \begin{pmatrix} x & g^T \\ 0 & P \end{pmatrix} . \begin{pmatrix} u_{\mathbf{h}} \\ u_{\mathbf{r}} \end{pmatrix}, \quad \text{where} \quad \begin{cases} x \in \mathbb{R},\ g \in \mathbb{R}^m,\ P \in \mathbb{R}^{m\times m} \\ \text{such that } xY = P^T.Y + Y.P \end{cases}$$

coincides with $\mathcal{H}^0_m(Y, \delta)$. For such a triple, the non-zero evaluations of the Riemannian curvature tensor $R_{\mathrm{o}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and the Ricci tensor $\mathrm{Ric}_{\mathrm{o}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ are determined by

$$\begin{aligned} R_{\mathrm{o}}(u, z) &= +\frac{1}{4}\mathcal{P}\left(\left(2(P + P^T).(P - xI_m) + (Y + P^T - P).(Y + P + P^T) - 2P^T.Y\right).u_{\mathbf{r}}\right) \\ \mathrm{Ric}_{\mathrm{o}}(z, z) &= -\frac{1}{4}\,\mathrm{tr}\,\left(2(P + P^T).(P - xI_m) + Y.Y\right) \end{aligned}$$

for all $u \in \mathfrak{g}$. ◇

The following corollary specializes on the geometry of the homogeneous triple $\mathcal{H}^n_0(\delta)$. To the best of our knowledge, any homogeneous triple published with geometry (4.3) prior to our thesis is of type $\mathcal{H}^n_0(\delta)$. For a complete discussion see [FM05], who also state the left-action on the corresponding homogeneous space.

Corollary 4.12. $\boxed{\mathcal{H}_0^n(\delta)}$ Let $\mathfrak{n}$ be the $(1+2n)$-dimensional Lie algebra with basis $\langle p_1, \ldots, p_n,$ $h,\ q_1, \ldots, q_n\rangle$ and commutator determined by

$$[u_{\mathfrak{n}}, v_{\mathfrak{n}}] = \begin{pmatrix} v_{\mathbf{p}} \\ v_{\mathbf{h}} \\ v_{\mathbf{q}} \end{pmatrix}^T \cdot \begin{pmatrix} 0 & 0 & -I_n \\ 0 & 0 & 0 \\ I_n & 0 & Z \end{pmatrix} \cdot \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \end{pmatrix} h \quad \text{for all } u, v \in \mathfrak{n},$$

where $Z \in \mathbb{R}^{n\times n}$ is a skew symmetric matrix. The homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ with Lie algebra $\mathfrak{g} = \mathfrak{n} \rtimes_\delta \mathbb{R}$, isotropy $\mathfrak{h} = \langle p_1, \ldots, p_n\rangle$, scalar product $B = L_{2+n}$ with respect to $\mathfrak{m} = \langle h, q_1, \ldots, q_n, z\rangle$, and derivation $\delta : \mathfrak{n} \to \mathfrak{n}$ as

$$u_{\mathfrak{n}} \mapsto \begin{pmatrix} 0 & 0 & S - xZ/2 \\ 0 & x & 0 \\ I_n & 0 & xI_n + Z \end{pmatrix} \cdot \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \end{pmatrix}, \qquad \text{where } x \in \mathbb{R}, \text{ and } S \in \mathbb{R}^{n\times n} \text{ symmetric}$$

coincides with $\mathcal{H}_0^n(\delta)$. For such a triple, the non-zero evaluations of the Riemannian curvature tensor $R_o : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ and the Ricci tensor $\mathrm{Ric}_o : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ are determined by

$$R_o(u, z) = +\mathcal{P}\left(\left(S + Z^2/4\right).u_{\mathbf{q}}\right), \quad \text{and} \quad \mathrm{Ric}_o(z, z) = -\operatorname{tr}\left(S + Z^2/4\right) \quad \text{for all } u \in \mathfrak{m}.$$

$\diamond$

In case $x, Z = 0$, the triple $\mathcal{H}_0^n(\delta)$ is moreover symmetric. According to [CW70], any symmetric triple with geometry (4.3) is of this form. We have

$$R_o = 0 \;\Leftrightarrow\; S = 0, \quad \text{and} \quad \mathrm{Ric}_o = 0 \;\Leftrightarrow\; \operatorname{tr} S = 0,$$

which is also well known from [Ne02].

4.3 Isomorphy

For integers $n, m \geq 0$, we denote with $\mathcal{H}_m^n$ the set of all homogeneous triples $\mathcal{H}_m^n(Y, \delta)$ that are in the scope of Definition 4.1. Isomorphy of homogeneous triples is an equivalence relation. A moduli space such as

$$\mathcal{M}_m^n := \mathcal{H}_m^n/\text{isomorphy}$$

typically has complicated topology. However, the next lemma is an important utility to classify the triples in $\mathcal{H}_m^n$.

Lemma 4.13. Consider the definitions in Table 4.1. The linear mappings

$$T^{\mathbf{h}}(\lambda), T^{\mathbf{q}}(Q), T^{\mathbf{r}}(R), T^{\mathbf{z}}(\eta) : \mathfrak{g} \to \mathfrak{g} \quad \text{for parameters } \lambda \in \mathbb{R}^*,\ Q \in \mathrm{O}(\mathbb{R}^n),\ R \in \mathrm{O}(\mathbb{R}^m),\ \eta \in \mathbb{R}^m$$

are isomorphisms between the homogeneous triples $\mathcal{H}_m^n(Y, \delta)$ and $\mathcal{H}_m^n(\check{Y}, \check{\delta})$.

Table 4.1 : We define the linear mappings $T^{\mathbf{h}}(\lambda), T^{\mathbf{q}}(Q), T^{\mathbf{r}}(R), T^{\mathbf{z}}(\eta) : \mathfrak{g} \to \mathfrak{g}$. Lemma 4.13 proves that the triples $\mathcal{H}_m^n(Y, \delta)$ and $\mathcal{H}_m^n(\breve{Y}, \breve{\delta})$ are isomorphic.

α	$\breve{\delta}$	$\breve{Y}$ $\breve{Z}$
$\mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$	$\delta = \begin{pmatrix} & & S - xZ/2 & -N^T.Y \\ & x & & g^T \\ I_n & & xI_n + Z & \\ & & N & P \end{pmatrix}$	Y Z
$T^{\mathbf{h}} : \mathbb{R}^* \to \mathrm{GL}(\mathfrak{g}), \lambda \mapsto$ $\begin{pmatrix} \lambda I_n & & & & \\ & \lambda & & & \\ & & I_n & & \\ & & & I_m & \\ & & & & 1/\lambda \end{pmatrix}$	$\begin{pmatrix} & & (S - xZ/2)/\lambda^2 & -N^T.Y/\lambda^2 \\ & x/\lambda & & g^T/\lambda^2 \\ I_n & & (xI_n + Z)/\lambda & \\ & & N/\lambda & P/\lambda \end{pmatrix}$	Y/λ Z/λ
$T^{\mathbf{q}} : \mathrm{O}(\mathbb{R}^n) \to \mathrm{GL}(\mathfrak{g}), Q \mapsto$ $\begin{pmatrix} Q & & & & \\ & 1 & & & \\ & & Q & & \\ & & & I_m & \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} & & Q^T.(S - xZ/2).Q & -Q^T.N^T.Y \\ & x & & g^T \\ I_n & & xI_n + Q^T.Z.Q & \\ & & N.Q & P \end{pmatrix}$	Y $Q^T.Z.Q$
$T^{\mathbf{r}} : \mathrm{O}(\mathbb{R}^m) \to \mathrm{GL}(\mathfrak{g}), R \mapsto$ $\begin{pmatrix} I_n & & & & \\ & 1 & & & \\ & & I_n & & \\ & & & R & \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} & & S - xZ/2 & -N^T.Y.R \\ & x & & g^T.R \\ I_n & & xI_n + Z & \\ & & R^T.N & R^T.P.R \end{pmatrix}$	$R^T.Y.R$ Z
$T^{\mathbf{z}} : \mathbb{R}^m \to \mathrm{GL}(\mathfrak{g}), \eta \mapsto$ $\begin{pmatrix} I_n & & & & N^T.\eta \\ & 1 & & \eta^T & -\eta^T.\eta/2 \\ & & I_n & & \\ & & & I_m & -\eta \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} & & S - xZ/2 & -N^T.Y \\ & x & & g^T + \eta^T.(xI_m + Y - P) \\ I_n & & xI_n + Z & \\ & & N & P \end{pmatrix}$	Y Z

Proof. Denote with ad, and $\breve{\text{ad}}$ the commutators of $\mathcal{H}^n_m(Y,\delta)$, and $\mathcal{H}^n_m(\breve{Y},\breve{\delta})$ respectively. In the spirit of Remark 1.9, we show that both diagrams commute

$$\begin{array}{rccccc} \breve{\text{ad}}: & \mathfrak{g} & \times & \mathfrak{g} & \to & \mathfrak{g} \\ & \downarrow \alpha & & \downarrow \alpha & & \uparrow \alpha^{-1} \\ \text{ad}: & \mathfrak{g} & \times & \mathfrak{g} & \to & \mathfrak{g} \end{array}, \quad \text{and} \quad \begin{array}{rccccc} L_{2+n+m}: & \mathfrak{m} & \times & \mathfrak{m} & \to & \mathbb{R} \\ & \downarrow \alpha_{\mathfrak{m}} & & \downarrow \alpha_{\mathfrak{m}} & & \uparrow \text{Id} \\ L_{2+n+m}: & \mathfrak{m} & \times & \mathfrak{m} & \to & \mathbb{R} \end{array}$$

for $\alpha = T^{\mathbf{h}}(\lambda), T^{\mathbf{q}}(Q), T^{\mathbf{r}}(R), T^{\mathbf{z}}(\eta)$, and all parameters $\lambda \in \mathbb{R}^*$, $Q \in \mathrm{O}(\mathbb{R}^n)$, $R \in \mathrm{O}(\mathbb{R}^m)$, $\eta \in \mathbb{R}^m$.

- First, consider $\boxed{\alpha = T^{\mathbf{z}}(\eta)}$ with $\eta \in \mathbb{R}^m$. The linear mapping is defined in Table 4.1 as

$$\alpha = \begin{pmatrix} I_n & & & & N^T.\eta \\ & 1 & & \eta^T & -\eta^T.\eta/2 \\ & & I_n & & \\ & & & I_m & -\eta \\ & & & & 1 \end{pmatrix}, \quad \text{and} \quad \alpha^{-1} = \begin{pmatrix} I_n & & & & -N^T.\eta \\ & 1 & & -\eta^T & -\eta^T.\eta/2 \\ & & I_n & & \\ & & & I_m & \eta \\ & & & & 1 \end{pmatrix}.$$

We have to show $[u,v]_{\breve{\text{ad}}} = \alpha^{-1}[\alpha.u, \alpha.v]$ for all $u, v \in \mathfrak{g}$, where plain $[\cdot,\cdot]$ denotes the commutator with respect to ad. By linearity, the equation decomposes into

$$[u_{\mathfrak{n}}, v_{\mathfrak{n}}]_{\breve{\text{ad}}} = \alpha^{-1}[\alpha_{\mathfrak{n}}.u_{\mathfrak{n}}, \alpha_{\mathfrak{n}}.v_{\mathfrak{n}}] \tag{4.12}$$

$$[z, u_{\mathfrak{n}}]_{\breve{\text{ad}}} = \alpha^{-1}[\alpha(z), \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}] \tag{4.13}$$

for all $u, v \in \mathfrak{g}$. The commutator in (4.12) reduces to matrix multiplication,

$$\begin{aligned} &\alpha^{-1}[\alpha_{\mathfrak{n}}.u_{\mathfrak{n}}, \alpha_{\mathfrak{n}}.v_{\mathfrak{n}}] = \dots \\ &= \Big(v_{\mathfrak{n}}^T.\alpha_{\mathfrak{n}}^T. \begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & Y \end{pmatrix} . \begin{pmatrix} I_n & & & \\ & 1 & & \eta^T \\ & & I_n & \\ & & & I_m \end{pmatrix} .u_{\mathfrak{n}}\Big)\alpha(h) \\ &= \Big(v_{\mathfrak{n}}^T. \begin{pmatrix} I_n & & & \\ & 1 & & \\ & & I_n & \\ & \eta & & I_m \end{pmatrix} . \begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & Y \end{pmatrix} .u_{\mathfrak{n}}\Big)h = \Big(v_{\mathfrak{n}}^T. \begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & Y \end{pmatrix} .u_{\mathfrak{n}}\Big)h. \end{aligned}$$

The commutator in (4.13) is slightly more complicated. We have

$$\alpha^{-1}\left[\alpha z, \alpha u_{\mathfrak{n}}\right] = \alpha^{-1}\left[\begin{pmatrix} N^T.\eta \\ -\eta^T.\eta/2 \\ 0 \\ -\eta \\ 1 \end{pmatrix}, \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}\right] = \alpha^{-1}\left[z, \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}\right] + \alpha^{-1}\left[\begin{pmatrix} N^T.\eta \\ -\eta^T.\eta/2 \\ 0 \\ -\eta \\ 0 \end{pmatrix}, \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}\right].$$

The first summand $\alpha^{-1}\,[z, \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}]$ is

$$
\begin{aligned}
\ldots &= \alpha_{\mathfrak{n}}^{-1}.\delta.\alpha_{\mathfrak{n}}.u_{\mathfrak{n}} \\
&= \alpha_{\mathfrak{n}}^{-1}.\begin{pmatrix} & & S - xZ/2 & -N^T.Y \\ & x & & g^T \\ I_n & & xI_n + Z & \\ & & N & P \end{pmatrix} . \begin{pmatrix} I_n & & & \\ & 1 & & \eta^T \\ & & I_n & \\ & & & I_m \end{pmatrix} .u_{\mathfrak{n}} \\
&= \begin{pmatrix} I_n & & & \\ & 1 & & -\eta^T \\ & & I_n & \\ & & & I_m \end{pmatrix} . \begin{pmatrix} & & S - xZ/2 & -N^T.Y \\ & x & & g^T + x\eta^T \\ I_n & & xI_n + Z & \\ & & N & P \end{pmatrix} .u_{\mathfrak{n}} \\
&= \begin{pmatrix} & & S - xZ/2 & -N^T.Y \\ & x & -\eta^T.N & g^T + x\eta^T - \eta^T.P \\ I_n & & xI_n + Z & \\ & & N & P \end{pmatrix} .u_{\mathfrak{n}},
\end{aligned}
$$

whereas the second summand simplifies to

$$
\begin{aligned}
\ldots = \alpha^{-1}\left[\alpha_{\mathfrak{n}}.u_{\mathfrak{n}}, \begin{pmatrix} -N^T.\eta \\ \eta^T.\eta/2 \\ 0 \\ \eta \\ 0 \end{pmatrix}\right] &= \left(\begin{pmatrix} -N^T.\eta \\ \eta^T.\eta/2 \\ 0 \\ \eta \end{pmatrix}^T . \begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & Y \end{pmatrix} .\alpha_{\mathfrak{n}}.u_{\mathfrak{n}} \right) h \\
&= \left(\begin{pmatrix} 0 \\ 0 \\ N^T.\eta \\ Y^T.\eta \end{pmatrix}^T . \begin{pmatrix} I_n & & & \\ & 1 & & \eta^T \\ & & I_n & \\ & & & I_m \end{pmatrix} .u_{\mathfrak{n}} \right) h = \begin{pmatrix} 0 & & & \\ & 0 & \eta^T.N & \eta^T.Y \\ & & 0 & \\ & & & 0 \end{pmatrix} .u_{\mathfrak{n}}.
\end{aligned}
$$

The results confirm $[z, u_{\mathfrak{n}}]_{\breve{\mathrm{ad}}} = \alpha^{-1}[\alpha(z), \alpha_{\mathfrak{n}}.u_{\mathfrak{n}}] = \breve{\delta}.u_{\mathfrak{n}}$ for all $u \in \mathfrak{g}$.

• For $\boxed{\alpha = T^{\mathbf{h}}(\lambda), T^{\mathfrak{q}}(Q), T^{\mathbf{r}}(R)}$ we suspend similar transformations to Table 4.9, and Table 4.10. However, the computations are carried out in detail.

The right diagram is a consequence of Remark 1.2. □

4.4 Auxiliary calculations

Table 4.2 : We yield the linear mapping $\xi:\mathfrak{n}\to\mathfrak{gl}(\mathfrak{n})$ defined by $\xi(v).u=\delta.[u,v]-[\delta.u,v]-[u,\delta.v]$ for $v\in\mathbf{p},\mathbf{q},\mathbf{r}$ respectively.

- $$\delta.[u,v_{\mathbf{p}}]=\delta.\left[u_{\mathbf{q}},v_{\mathbf{p}}\right]=\delta.\left(-(v_{\mathbf{p}}^T.u_{\mathbf{q}})h\right)=-\begin{pmatrix}b\\x\\i\\l\end{pmatrix}(v_{\mathbf{p}}^T.u_{\mathbf{q}})=-\begin{pmatrix}b.v_{\mathbf{p}}^T\\x.v_{\mathbf{p}}^T\\i.v_{\mathbf{p}}^T\\l.v_{\mathbf{p}}^T\end{pmatrix}.u_{\mathbf{q}}$$

$$-[\delta.u,v_{\mathbf{p}}]=-\left[\begin{pmatrix}A&b&C&D\\0&x&f^T&g^T\\I_n&i&J&K\\0&l&N&P\end{pmatrix}.\begin{pmatrix}u_{\mathbf{p}}\\u_h\\u_{\mathbf{q}}\\u_{\mathbf{r}}\end{pmatrix},v_{\mathbf{p}}\right]=\big(v_{\mathbf{p}}^T.\begin{pmatrix}I_n&i&J&K\end{pmatrix}.\begin{pmatrix}u_{\mathbf{p}}\\u_h\\u_{\mathbf{q}}\\u_{\mathbf{r}}\end{pmatrix}\big)h$$

$$-[u,\delta(v_{\mathbf{p}})]=-\left[\begin{pmatrix}u_{\mathbf{p}}\\u_h\\u_{\mathbf{q}}\\u_{\mathbf{r}}\end{pmatrix},\begin{pmatrix}A.v_{\mathbf{p}}\\0\\I_n.v_{\mathbf{p}}\\0\end{pmatrix}\right]=-\big(v_{\mathbf{p}}^T.I_n.u_{\mathbf{p}}-v_{\mathbf{p}}^T.A^T.u_{\mathbf{q}}\big)h$$

- $$\delta.[u,v_{\mathbf{q}}]=\delta.\left[u_{\mathbf{p}},v_{\mathbf{q}}\right]=\delta.\left((v_{\mathbf{q}}^T.u_{\mathbf{p}})h\right)=x\left(v_{\mathbf{q}}^T.u_{\mathbf{p}}\right)h$$

$$-[\delta.u,v_{\mathbf{q}}]=-\left[\begin{pmatrix}A&0&C&D\\0&x&f^T&g^T\\I_n&0&J&0\\0&0&N&P\end{pmatrix}.\begin{pmatrix}u_{\mathbf{p}}\\u_h\\u_{\mathbf{q}}\\u_{\mathbf{r}}\end{pmatrix},v_{\mathbf{q}}\right]=-\big(v_{\mathbf{q}}^T.\begin{pmatrix}A&0&C&D\end{pmatrix}.\begin{pmatrix}u_{\mathbf{p}}\\u_h\\u_{\mathbf{q}}\\u_{\mathbf{r}}\end{pmatrix}\big)h$$

$$-[u,\delta(v_{\mathbf{q}})]=-\left[\begin{pmatrix}u_{\mathbf{p}}\\u_h\\u_{\mathbf{q}}\\u_{\mathbf{r}}\end{pmatrix},\begin{pmatrix}C.v_{\mathbf{q}}\\f^T.v_{\mathbf{q}}\\J.v_{\mathbf{q}}\\N.v_{\mathbf{q}}\end{pmatrix}\right]=-\left(v_{\mathbf{q}}^T.J^T.u_{\mathbf{p}}-v_{\mathbf{q}}^T.C^T.u_{\mathbf{q}}+v_{\mathbf{q}}^T.N^T.Y.u_{\mathbf{r}}\right)h$$

- $$\delta.[u,v_{\mathbf{r}}]=\delta.\left[u_{\mathbf{r}},v_{\mathbf{r}}\right]=\delta.\left((v_{\mathbf{r}}^T.Y.u_{\mathbf{r}})h\right)=\left(v_{\mathbf{r}}^T.xY.u_{\mathbf{r}}\right)h$$

$$-[\delta.u,v_{\mathbf{r}}]=-\left[\begin{pmatrix}A&0&C&D\\0&x&f^T&g^T\\I_n&0&J&0\\0&0&N&P\end{pmatrix}.\begin{pmatrix}u_{\mathbf{p}}\\u_h\\u_{\mathbf{q}}\\u_{\mathbf{r}}\end{pmatrix},v_{\mathbf{r}}\right]=-\big(v_{\mathbf{r}}^T.Y.\begin{pmatrix}0&0&N&P\end{pmatrix}.\begin{pmatrix}u_{\mathbf{p}}\\u_h\\u_{\mathbf{q}}\\u_{\mathbf{r}}\end{pmatrix}\big)h$$

$$-[u,\delta(v_{\mathbf{r}})]=-\left[\begin{pmatrix}u_{\mathbf{p}}\\u_h\\u_{\mathbf{q}}\\u_{\mathbf{r}}\end{pmatrix},\begin{pmatrix}D.v_{\mathbf{q}}\\g^T.v_{\mathbf{q}}\\0.v_{\mathbf{q}}\\P.v_{\mathbf{q}}\end{pmatrix}\right]=-\big(-v_{\mathbf{q}}^T.D^T.u_{\mathbf{q}}+v_{\mathbf{q}}^T.P^T.Y.u_{\mathbf{r}}\big)h$$

Table 4.3 : First, we write out the commutator $[w,u_{\mathfrak{n}}]_{\mathfrak{m}}$. Then, we obtain $\nu(u_{\mathfrak{n}},v_{\mathfrak{n}})$ via the relation $2B\big(\nu(u,v),w_{\mathfrak{m}}\big)=B\big(u_{\mathfrak{m}},[w,v]_{\mathfrak{m}}\big)+B\big([w,u]_{\mathfrak{m}},v_{\mathfrak{m}}\big)$ for all $u,v,w\in\mathfrak{g}$.

$$\begin{aligned}
[w,u_{\mathfrak{n}}]_{\mathfrak{m}} =&[w_{\mathfrak{n}},u_{\mathfrak{n}}]_{\mathfrak{m}}+w_z(\delta.u_{\mathfrak{n}})_{\mathfrak{m}}\\
=&\begin{pmatrix} u_{\mathbf{p}}\\ u_{\mathbf{h}}\\ u_{\mathbf{q}}\\ u_{\mathbf{r}}\end{pmatrix}^T.\begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & Y\end{pmatrix}.\begin{pmatrix} w_{\mathbf{p}}\\ w_{\mathbf{h}}\\ w_{\mathbf{q}}\\ w_{\mathbf{r}}\end{pmatrix}h+w_z\begin{pmatrix} & & x & g\\ I_n & & xI_n+Z & \\ & & N & P\\ 0 & 0 & 0 & 0\end{pmatrix}.\begin{pmatrix} u_{\mathbf{p}}\\ u_{\mathbf{h}}\\ u_{\mathbf{q}}\\ u_{\mathbf{r}}\end{pmatrix}\\
=&\begin{pmatrix} u_{\mathbf{q}}^T & 0 & u_{\mathbf{q}}^T.Z-u_{\mathbf{p}}^T & u_{\mathbf{r}}^T.Y & xu_{\mathbf{h}}+u_{\mathbf{r}}^T.g\\ 0 & 0 & 0 & 0 & u_{\mathbf{p}}+(xI_n+Z).u_{\mathbf{q}}\\ 0 & 0 & 0 & 0 & N.u_{\mathbf{q}}+P.u_{\mathbf{r}}\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.\begin{pmatrix} w_{\mathbf{p}}\\ w_{\mathbf{h}}\\ w_{\mathbf{q}}\\ w_{\mathbf{r}}\\ w_{\mathbf{z}}\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
2B\big(\nu(u_{\mathfrak{n}},v_{\mathfrak{n}}),w_{\mathfrak{m}}\big) =&B\big((u_{\mathfrak{n}})_{\mathfrak{m}},[w_{\mathfrak{m}},v_{\mathfrak{n}}]_{\mathfrak{m}}\big)+B\big([w_{\mathfrak{m}},u_{\mathfrak{n}}]_{\mathfrak{m}},(v_{\mathfrak{n}})_{\mathfrak{m}}\big)\\
=+&\begin{pmatrix} u_{\mathbf{h}}\\ u_{\mathbf{q}}\\ u_{\mathbf{r}}\\ 0\end{pmatrix}^T.L_{n+m}.\begin{pmatrix} 0 & v_{\mathbf{q}}^T.Z-v_{\mathbf{p}}^T & v_{\mathbf{r}}^T.Y & xv_{\mathbf{h}}+g^T.v_{\mathbf{r}}\\ 0 & 0 & 0 & v_{\mathbf{p}}+(xI_n+Z).v_{\mathbf{q}}\\ 0 & 0 & 0 & N.v_{\mathbf{q}}+P.v_{\mathbf{r}}\\ 0 & 0 & 0 & 0\end{pmatrix}.\begin{pmatrix} w_{\mathbf{h}}\\ w_{\mathbf{q}}\\ w_{\mathbf{r}}\\ w_{\mathbf{z}}\end{pmatrix}\\
+&\begin{pmatrix} v_{\mathbf{h}}\\ v_{\mathbf{q}}\\ v_{\mathbf{r}}\\ 0\end{pmatrix}^T.L_{n+m}.\begin{pmatrix} 0 & u_{\mathbf{q}}^T.Z-u_{\mathbf{p}}^T & u_{\mathbf{r}}^T.Y & xu_{\mathbf{h}}+g^T.u_{\mathbf{r}}\\ 0 & 0 & 0 & u_{\mathbf{p}}+(xI_n+Z).u_{\mathbf{q}}\\ 0 & 0 & 0 & N.u_{\mathbf{q}}+P.u_{\mathbf{r}}\\ 0 & 0 & 0 & 0\end{pmatrix}.\begin{pmatrix} w_{\mathbf{h}}\\ w_{\mathbf{q}}\\ w_{\mathbf{r}}\\ w_{\mathbf{z}}\end{pmatrix}\\
=+&\begin{pmatrix} u_{\mathbf{h}}\\ u_{\mathbf{q}}\\ u_{\mathbf{r}}\\ 0\end{pmatrix}^T.\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & v_{\mathbf{p}}+(xI_n+Z).v_{\mathbf{q}}\\ 0 & 0 & 0 & N.v_{\mathbf{q}}+P.v_{\mathbf{r}}\\ 0 & 0 & 0 & 0\end{pmatrix}.\begin{pmatrix} w_{\mathbf{h}}\\ w_{\mathbf{q}}\\ w_{\mathbf{r}}\\ w_{\mathbf{z}}\end{pmatrix}\\
+&\begin{pmatrix} v_{\mathbf{h}}\\ v_{\mathbf{q}}\\ v_{\mathbf{r}}\\ 0\end{pmatrix}^T.\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & u_{\mathbf{p}}+(xI_n+Z).u_{\mathbf{q}}\\ 0 & 0 & 0 & N.u_{\mathbf{q}}+P.u_{\mathbf{r}}\\ 0 & 0 & 0 & 0\end{pmatrix}.\begin{pmatrix} w_{\mathbf{h}}\\ w_{\mathbf{q}}\\ w_{\mathbf{r}}\\ w_{\mathbf{z}}\end{pmatrix}
\end{aligned}$$

$$B\big(\nu(u_{\mathfrak{n}},v_{\mathfrak{n}}),w_{\mathfrak{m}}\big)=\begin{pmatrix} v_{\mathbf{p}}\\ v_{\mathbf{h}}\\ v_{\mathbf{q}}\\ v_{\mathbf{r}}\end{pmatrix}^T.\begin{pmatrix} 0 & 0 & I_n/2 & 0\\ 0 & 0 & 0 & 0\\ I_n/2 & 0 & xI_n & N^T/2\\ 0 & 0 & N/2 & (P+P^T)/2\end{pmatrix}.\begin{pmatrix} u_{\mathbf{p}}\\ u_{\mathbf{h}}\\ u_{\mathbf{q}}\\ u_{\mathbf{r}}\end{pmatrix}w_{\mathbf{z}}$$

Table 4.4 : First, we write out the commutator $[w,z]_{\mathfrak{m}}$. Then, we obtain $\nu(u,z)$ via the relation $2B\big(\nu(u,z),w_{\mathfrak{m}}\big) = B(u_{\mathfrak{m}},[w,z]_{\mathfrak{m}}) + B(z,[w,u]_{\mathfrak{m}})$ for all $u,w \in \mathfrak{g}$.

$$[w,z]_{\mathfrak{m}} = -\,[z,w]_{\mathfrak{m}} = -(\delta.w_{\mathfrak{n}})_{\mathfrak{m}} = -\begin{pmatrix} 0 & x & 0 & g^T & 0 \\ I_n & 0 & x+Z & 0 & 0 \\ 0 & 0 & N & P & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_{\mathbf{p}} \\ w_{\mathbf{h}} \\ w_{\mathbf{q}} \\ w_{\mathbf{r}} \\ w_{\mathbf{z}} \end{pmatrix}$$

$$\begin{aligned}
2B\big(\nu(u,z),w_{\mathfrak{m}}\big) =& B(u_{\mathfrak{m}},[w,z]_{\mathfrak{m}}) + B(z,[w,u]_{\mathfrak{m}}) \\
=& B(u_{\mathfrak{m}},[w,z]_{\mathfrak{m}}) + u_{\mathbf{z}}B(z,[w,z]_{\mathfrak{m}}) + B(z,[w,u_{\mathfrak{n}}]_{\mathfrak{m}}) \\
=& B(u_{\mathfrak{m}}+u_{\mathbf{z}},[w,z]_{\mathfrak{m}}) + B(z,[w,u_{\mathfrak{n}}]_{\mathfrak{m}}) \\
=& B(u_{\mathfrak{m}}+u_{\mathbf{z}},[w,z]_{\mathfrak{m}}) + B(z,[w,u_{\mathfrak{n}}]_{\mathbf{h}}) \\
=& [w,u_{\mathfrak{n}}]_{\mathbf{h}} + B(u_{\mathfrak{m}}+u_{\mathbf{z}},[w,z]_{\mathfrak{m}}) \\
=& + \begin{pmatrix} u_{\mathbf{q}}^T & 0 & u_{\mathbf{q}}^T.Z - u_{\mathbf{p}}^T & u_{\mathbf{r}}^T.Y & xu_{\mathbf{h}} + u_{\mathbf{r}}^T.g \end{pmatrix}.w \\
& - \begin{pmatrix} u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \\ 2u_{\mathbf{z}} \end{pmatrix}^T .L_{n+m}. \begin{pmatrix} 0 & x & 0 & g^T & 0 \\ I_n & 0 & xI_n+Z & 0 & 0 \\ 0 & 0 & N & P & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.w \\
=& \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \\ u_{\mathbf{z}} \end{pmatrix}^T \cdot \left(\begin{pmatrix} 0 & 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & x \\ I_n & 0 & Z & 0 & 0 \\ 0 & 0 & 0 & Y & g \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & xI_n+Z & 0 & 0 \\ 0 & 0 & N & P & 0 \\ 0 & 2x & 0 & 2g^T & 0 \end{pmatrix} \right).w
\end{aligned}$$

$$\nu(u,z)^T.B.w_{\mathfrak{m}} = \frac{1}{2} \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \\ u_{\mathbf{z}} \end{pmatrix}^T \cdot \begin{pmatrix} 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & -xI_n & 0 & 0 \\ 0 & -N & Y-P & g \\ -2x & 0 & -2g^T & 0 \end{pmatrix}.w_{\mathfrak{m}}$$

$$\nu(u,z) = \frac{1}{2}L_{n+m+2}. \begin{pmatrix} 0 & 0 & 0 & 0 & -2x \\ -I_n & 0 & -xI_n & -N & 0 \\ 0 & 0 & 0 & Y-P & -2g \\ 0 & x & 0 & g^T & 0 \end{pmatrix} \cdot \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \\ u_{\mathbf{z}} \end{pmatrix}$$

Table 4.5 : We derive the Levi-Civita connection $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$. It turns out, that $\Lambda(h) = 0$. The identities below hold for all $u, v \in \mathfrak{g}$.

$$\begin{aligned}
\Lambda(u_{\mathfrak{n}}).(v_{\mathfrak{n}})_{\mathfrak{m}} =& \frac{1}{2}[u_{\mathfrak{n}}, (v_{\mathfrak{n}})_{\mathfrak{m}}]_{\mathfrak{m}} + \nu(u_{\mathfrak{n}}, (v_{\mathfrak{n}})_{\mathfrak{m}}) \\
=& + \begin{pmatrix} 0 \\ v_{\mathbf{h}} \\ v_{\mathbf{q}} \\ v_{\mathbf{r}} \end{pmatrix}^T . \begin{pmatrix} 0 & -I_n/2 & \\ & 0 & \\ I_n/2 & Z/2 & \\ & & Y/2 \end{pmatrix} . \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix} h \\
& + \begin{pmatrix} 0 \\ v_{\mathbf{h}} \\ v_{\mathbf{q}} \\ v_{\mathbf{r}} \end{pmatrix}^T . \begin{pmatrix} 0 & 0 & I_n/2 & 0 \\ 0 & 0 & 0 & 0 \\ I_n/2 & 0 & xI_n & N^T/2 \\ 0 & 0 & N/2 & (P+P^T)/2 \end{pmatrix} . \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix} h \\
=& \begin{pmatrix} v_{\mathbf{h}} \\ v_{\mathbf{q}} \\ v_{\mathbf{r}} \\ v_{\mathbf{z}} \end{pmatrix}^T . \begin{pmatrix} 0 & 0 & 0 & 0 \\ I_n & 0 & xI_n + Z/2 & N^T/2 \\ 0 & 0 & N/2 & (P+P^T+Y)/2 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix} h
\end{aligned}$$

Due to the $\Lambda(u)$-invariance of $B = L_{n+m+2}$ for all $u \in \mathfrak{g}$, we obtain

$$\begin{aligned}
B((v_{\mathfrak{n}})_{\mathfrak{m}}, \Lambda(u_{\mathfrak{n}}).z) =& - B(z, \Lambda(u_{\mathfrak{n}}).(v_{\mathfrak{n}})_{\mathfrak{m}}) = -(\Lambda(u_{\mathfrak{n}}).(v_{\mathfrak{n}})_{\mathfrak{m}})_{\mathbf{h}} \\
(v_{\mathfrak{n}})_{\mathfrak{m}}^T.B.\Lambda(u_{\mathfrak{n}}).z =& - \Lambda(u_{\mathfrak{n}}).(v_{\mathfrak{n}})_{\mathfrak{m}} \\
\Lambda(u_{\mathfrak{n}}).z =& - \begin{pmatrix} 0 & 0 & 0 & 0 \\ I_n & 0 & xI_n + Z/2 & N^T/2 \\ 0 & 0 & N/2 & (P+P^T+Y)/2 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix}
\end{aligned}$$

It remains to derive the linear mapping $\Lambda(z)$:

$$\begin{aligned}
\Lambda(z).v_{\mathfrak{m}} =& \frac{1}{2}[z, v_{\mathfrak{m}}]_{\mathfrak{m}} + \nu(z, v_{\mathfrak{m}}) = \frac{1}{2}(\delta.(v_{\mathfrak{m}})_{\mathfrak{n}})_{\mathfrak{m}} + \nu(v_{\mathfrak{m}}, z) \\
=& (\begin{pmatrix} x/2 & 0 & g^T/2 & 0 \\ 0 & xI_n/2 + Z/2 & 0 & 0 \\ 0 & N/2 & P/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} x/2 & 0 & g^T/2 & 0 \\ 0 & -xI_n/2 & -N^T/2 & 0 \\ 0 & 0 & -(Y+P^T)/2 & -g \\ 0 & 0 & 0 & -x \end{pmatrix}).v_{\mathfrak{m}}
\end{aligned}$$

Table 4.6 : We yield the essential evaluation of the Riemannian curvature $R_{\mathrm{o}}(u_{\mathfrak{m}\backslash z}, z)$ for all $u \in \mathfrak{m}$. First, we specialize the expressions of the Levi-Civita connection to

$$\Lambda(u_{\mathfrak{m}\backslash z}) = \mathcal{P}\left(\begin{pmatrix} xI_n + Z/2 & N^T/2 \\ N/2 & (P + P^T + Y)/2 \end{pmatrix} . \begin{pmatrix} u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix}\right)$$
$$\Lambda(z) = \mathcal{Q}\left(x, \begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} Z/2 & -N^T/2 \\ N/2 & (P - Y - P^T)/2 \end{pmatrix}\right).$$

This allows us to write

$$\begin{aligned} R_{\mathrm{o}}(u_{\mathfrak{m}\backslash z}, z) =& \\ =& \big[\Lambda(u_{\mathfrak{m}\backslash z}), \Lambda(z)\big]_{\mathfrak{gl}} - \Lambda\big([u_{\mathfrak{m}\backslash z}, z]\big) \\ =& \big[\Lambda(u_{\mathfrak{m}\backslash z}), \Lambda(z)\big]_{\mathfrak{gl}} + \Lambda.\delta(u_{\mathfrak{m}\backslash z}) \\ =& -\mathcal{P}\left(\begin{pmatrix} xI_n + Z/2 & -N^T/2 \\ N/2 & xI_m + (P - Y - P^T)/2 \end{pmatrix} . \begin{pmatrix} xI_n + Z/2 & N^T/2 \\ N/2 & (P + P^T + Y)/2 \end{pmatrix} . \begin{pmatrix} u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix}\right) \\ &+ \mathcal{P}\left(\begin{pmatrix} I_n & 0 & xI_n + Z/2 & N^T/2 \\ 0 & 0 & N/2 & (P + P^T + Y)/2 \end{pmatrix} . \begin{pmatrix} 0 & S - xZ/2 & -N^T.Y \\ x & 0 & g^T \\ 0 & xI_n + Z & 0 \\ 0 & N & P \end{pmatrix} . \begin{pmatrix} u_{\mathbf{h}} \\ u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix}\right). \end{aligned}$$

The matrix products in the previous sum expand to

$$\begin{aligned} A_1 :=& 4\begin{pmatrix} xI_n + Z/2 & -N^T/2 \\ N/2 & xI_m + (P - Y - P^T)/2 \end{pmatrix} . \begin{pmatrix} xI_n + Z/2 & N^T/2 \\ N/2 & (P + P^T + Y)/2 \end{pmatrix} \\ =& \begin{pmatrix} 4(xI_n + Z/2)^2 - N^T.N & 2(xI_n + Z/2)N^T - N^T(P + P^T + Y) \\ 2N(xI_n + Z/2) + 2(xI_m + (P - Y - P^T)/2).N & T_{\mathbf{r},\mathbf{r}} \end{pmatrix}, \end{aligned}$$

where $T_{\mathbf{r},\mathbf{r}} = N.N^T + 2(xI_m + (P - Y - P^T)/2)(P + P^T + Y)$, and

$$\begin{aligned} A_2 :=& 4\begin{pmatrix} I_n & 0 & xI_n + Z/2 & N^T/2 \\ 0 & 0 & N/2 & (P + P^T + Y)/2 \end{pmatrix} . \begin{pmatrix} S - xZ/2 & -N^T.Y \\ 0 & g^T \\ xI_n + Z & 0 \\ N & P \end{pmatrix} \\ =& \begin{pmatrix} 4S - 2xZ + 4(xI_n + Z/2).(xI_n + Z) + 2N^T.N & -4N^T.Y + 2N^T.P \\ 2N.(xI_n + Z) + 2(P + P^T + Y).N & 2(P + P^T + Y).P \end{pmatrix}. \end{aligned}$$

Then, the expression we are looking for is

$$R_{\mathrm{o}}(u_{\mathfrak{m}\backslash z}, z) = \frac{1}{4}\mathcal{P}\left(\big(-A_1 + A_2\big) . \begin{pmatrix} u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix}\right).$$

Table 4.7 : We have defined the matrices A_1 and A_2 in Table 4.6. A_1 and A_2 determine the Riemannian curvature tensor. Below, we simplify each of the four components in the expression $-A_1 + A_2$ with respect to the decomposition

$$-A_1 + A_2 = \begin{pmatrix} C_{\mathbf{q},\mathbf{q}} & C_{\mathbf{q},\mathbf{r}} \\ C_{\mathbf{r},\mathbf{q}} & C_{\mathbf{r},\mathbf{r}} \end{pmatrix}.$$

Combining the components of A_1 and A_2 by pairs, we obtain

$$\begin{aligned}
C_{\mathbf{q},\mathbf{q}} =& -4(xI_n + Z/2)^2 + N^T.N + 4S - 2xZ + 4(xI_n + Z/2).(xI_n + Z) + 2N^T.N \\
=& 4S + 3N^T.N + 4(xI_n + Z/2)(xI_n + Z - xI_n - Z/2) - 2xZ \\
=& 4S + 3N^T.N + 2(xI_n + Z/2)Z - 2xZ \\
=& 4S + 3N^T.N + Z.Z \\
C_{\mathbf{q},\mathbf{r}} =& -2(xI_n + Z/2)N^T + N^T(P + P^T + Y) - 4N^T.Y + 2N^T.P \\
=& -2(xI_n + Z/2)N^T + N^T(P + P^T + Y - 4Y + 2P) \\
=& N^T.(3P + P^T - 3Y) - (2xI_n + Z).N^T \\
C_{\mathbf{r},\mathbf{r}} =& -N.N^T - 2(xI_m + (P - Y - P^T)/2).(P + P^T + Y) + 2(P + P^T + Y).P \\
=& -(2xI_m + P - Y - P^T).(Y + P + P^T) + 2(Y + P + P^T).P - N.N^T \\
=& -2xY - 2x(P + P^T) - (P - Y - P^T).(Y + P + P^T) + 2(Y + P + P^T).P - N.N^T \\
=& -2P^T.Y - 2Y.P - 2x(P + P^T) + 2Y.P + 2(P + P^T).P \\
& -(P - Y - P^T).(Y + P + P^T) - N.N^T \\
=& 2(P + P^T).(P - xI_m) + (Y + P^T - P).(Y + P + P^T) - 2P^T.Y - N.N^T \\
C_{\mathbf{r},\mathbf{q}} =& C_{\mathbf{q},\mathbf{r}}^T = (3P^T + P + 3Y).N - N.(2xI_n - Z)
\end{aligned}$$

Table 4.8 : Recall, the trace of the product of a symmetric and a skew symmetric matrix is zero. Keeping this in mind, we simplify the expressions that determine the Ricci curvature.

$$\begin{aligned}
\operatorname{tr} C_{\mathbf{q},\mathbf{q}} &= \operatorname{tr}\left(4S + 3N^T.N + Z^2\right) \\
\operatorname{tr} C_{\mathbf{r},\mathbf{r}} &= \operatorname{tr}\left(2(P + P^T).(P - xI_m) + (Y + P^T - P).(Y + P + P^T) - 2P^T.Y - N.N^T\right) \\
&= \operatorname{tr}\left(2(P + P^T).(P - xI_m) + (Y + P^T - P).Y - 2P^T.Y - N.N^T\right) \\
&= \operatorname{tr}\left(2(P + P^T).(P - xI_m) + Y.Y - (P^T + P).Y - N.N^T\right) \\
&= \operatorname{tr}\left(2(P + P^T).(P - xI_m) + Y.Y - N.N^T\right)
\end{aligned}$$

Table 4.9 : Denote with $X, \breve{X}$ the matrices defined by $[u_{\mathfrak{n}}, v_{\mathfrak{n}}]_{\mathrm{ad}} = \big(v_{\mathfrak{n}}^T.X.u_{\mathfrak{n}}\big)h$, and $[u_{\mathfrak{n}}, v_{\mathfrak{n}}]_{\breve{\mathrm{ad}}} = \big(v_{\mathfrak{n}}^T.\breve{X}.u_{\mathfrak{n}}\big)h$ respectively. We observe that the relation $[u_{\mathfrak{n}}, v_{\mathfrak{n}}]_{\breve{\mathrm{ad}}} = \alpha^{-1}\,[\alpha_{\mathfrak{n}}.u_{\mathfrak{n}}, \alpha_{\mathfrak{n}}.v_{\mathfrak{n}}]$ for all $u, v \in \mathfrak{g}$ reduces to the equation $\big(\alpha_{\mathfrak{n}}^T.X.\alpha_{\mathfrak{n}}\big)\alpha^{-1}(h) = \breve{X}h$.

Let $\boxed{\alpha = T^{\mathbf{h}}(\lambda)}$ and $\lambda \in \mathbb{R}^*$. We confirm

$$\alpha_{\mathfrak{n}}^T.X.\alpha_{\mathfrak{n}}/\lambda = \alpha_{\mathfrak{n}}^T.\begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & Y \end{pmatrix}.\begin{pmatrix} \lambda I_n & & & \\ & \lambda & & \\ & & I_n & \\ & & & I_m \end{pmatrix}/\lambda$$

$$= \begin{pmatrix} \lambda I_n & & & \\ & \lambda & & \\ & & I_n & \\ & & & I_m \end{pmatrix}.\begin{pmatrix} 0 & & -I_n/\lambda & \\ & 0 & & \\ I_n & & Z/\lambda & \\ & & & Y/\lambda \end{pmatrix} = \begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z/\lambda & \\ & & & Y/\lambda \end{pmatrix} = \breve{X}.$$

Let $\boxed{\alpha = T^{\mathbf{q}}(Q)}$ and $Q \in \mathrm{O}(\mathbb{R}^n)$. We confirm

$$\alpha_{\mathfrak{n}}^T.X.\alpha_{\mathfrak{n}} = \alpha_{\mathfrak{n}}^T.\begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & Y \end{pmatrix}.\begin{pmatrix} Q & & & \\ & 1 & & \\ & & Q & \\ & & & I_m \end{pmatrix}$$

$$= \begin{pmatrix} Q^T & & & \\ & 1 & & \\ & & Q^T & \\ & & & I_m \end{pmatrix}.\begin{pmatrix} 0 & & -Q & \\ & 0 & & \\ Q & & Z.Q & \\ & & & Y \end{pmatrix} = \begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Q^T.Z.Q & \\ & & & Y \end{pmatrix} = \breve{X}.$$

Let $\boxed{\alpha = T^{\mathbf{r}}(R)}$ and $R \in \mathrm{O}(\mathbb{R}^m)$. We confirm

$$\alpha_{\mathfrak{n}}^T.X.\alpha_{\mathfrak{n}} = \alpha_{\mathfrak{n}}^T.\begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & Y \end{pmatrix}.\begin{pmatrix} I_n & & & \\ & 1 & & \\ & & I_n & \\ & & & R \end{pmatrix}$$

$$= \begin{pmatrix} I_n & & & \\ & 1 & & \\ & & I_n & \\ & & & R^T \end{pmatrix}.\begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & Y.R \end{pmatrix} = \begin{pmatrix} 0 & & -I_n & \\ & 0 & & \\ I_n & & Z & \\ & & & R^T.Y.R \end{pmatrix} = \breve{X}.$$

Table 4.10 : Finally, we confirm that $[z, u_{\mathfrak{n}}]_{\breve{\mathrm{ad}}} = \alpha^{-1}[\alpha z, \alpha u_{\mathfrak{n}}]$ for all $u \in \mathfrak{g}$. In all instances $\alpha = T^{\mathbf{z}}(\lambda), T^{\mathbf{q}}(Q), T^{\mathbf{r}}(R)$, this relation is equivalent to a simple matrix equation.

Let $\boxed{\alpha = T^{\mathbf{h}}(\lambda)}$ and $\lambda \in \mathbb{R}^*$. Then, $[z, u_{\mathfrak{n}}]_{\breve{\mathrm{ad}}} = \alpha^{-1}[\alpha z, \alpha u_{\mathfrak{n}}]$ reduces to $\alpha_{\mathfrak{n}}^{-1}.\delta.\alpha_{\mathfrak{n}}/\lambda = \breve{\delta}$, which we verify as:

$$\begin{pmatrix} I_n/\lambda & & & \\ & 1/\lambda & & \\ & & I_n & \\ & & & I_m \end{pmatrix} . \begin{pmatrix} & & S - xZ/2 & -N^T.Y \\ & x & & g^T \\ I_n & & xI_n + Z & \\ & & N & P \end{pmatrix} . \begin{pmatrix} \lambda I_n & & & \\ & \lambda & & \\ & & I_n & \\ & & & I_m \end{pmatrix} /\lambda$$

$$= \alpha_{\mathfrak{n}}^{-1} . \begin{pmatrix} & & (S - xZ/2)/\lambda & -N^T.Y/\lambda \\ & x & & g^T/\lambda \\ I_n & & (xI_n + Z)/\lambda & \\ & & N/\lambda & P/\lambda \end{pmatrix} = \begin{pmatrix} & & (S - xZ/2)/\lambda^2 & -N^T.Y/\lambda^2 \\ & x/\lambda & & g^T/\lambda^2 \\ I_n & & (xI_n + Z)/\lambda & \\ & & N/\lambda & P/\lambda \end{pmatrix}$$

Let $\boxed{\alpha = T^{\mathbf{q}}(Q)}$ and $Q \in \mathrm{O}(\mathbb{R}^n)$. Then, $[z, u_{\mathfrak{n}}]_{\breve{\mathrm{ad}}} = \alpha^{-1}[\alpha z, \alpha u_{\mathfrak{n}}]$ reduces to $\alpha_{\mathfrak{n}}^{-1}.\delta.\alpha_{\mathfrak{n}} = \breve{\delta}$, which we verify as:

$$\begin{pmatrix} Q^T & & & \\ & 1 & & \\ & & Q^T & \\ & & & I_m \end{pmatrix} . \begin{pmatrix} & & S - xZ/2 & -N^T.Y \\ & x & & g^T \\ I_n & & xI_n + Z & \\ & & N & P \end{pmatrix} . \begin{pmatrix} Q & & & \\ & 1 & & \\ & & Q & \\ & & & I_m \end{pmatrix}$$

$$= \alpha_{\mathfrak{n}}^{-1} . \begin{pmatrix} & & (S - xZ/2).Q & -N^T.Y \\ & x & & g^T \\ Q & & (xI_n + Z).Q & \\ & & N.Q & P \end{pmatrix} = \begin{pmatrix} & & Q^T.(S - xZ/2).Q & -Q^T.N^T.Y \\ & x & & g^T \\ I_n & & Q^T.(xI_n + Z).Q & \\ & & N.Q & P \end{pmatrix}$$

Let $\boxed{\alpha = T^{\mathbf{r}}(R)}$ and $R \in \mathrm{O}(\mathbb{R}^m)$. Then, $[z, u_{\mathfrak{n}}]_{\breve{\mathrm{ad}}} = \alpha^{-1}[\alpha z, \alpha u_{\mathfrak{n}}]$ reduces to $\alpha_{\mathfrak{n}}^{-1}.\delta.\alpha_{\mathfrak{n}} = \breve{\delta}$, which we verify as:

$$\begin{pmatrix} I_n & & & \\ & 1 & & \\ & & I_n & \\ & & & R^T \end{pmatrix} . \begin{pmatrix} & & S - xZ/2 & -N^T.Y \\ & x & & g^T \\ I_n & & xI_n + Z & \\ & & N & P \end{pmatrix} . \begin{pmatrix} I_n & & & \\ & 1 & & \\ & & I_n & \\ & & & R \end{pmatrix}$$

$$= \alpha_{\mathfrak{n}}^{-1} . \begin{pmatrix} & & S - xZ/2 & -N^T.Y.R \\ & x & & g^T.R \\ I_n & & xI_n + Z & \\ & & N & P.R \end{pmatrix} = \begin{pmatrix} & & S - xZ/2 & -N^T.Y.R \\ & x & & g^T.R \\ I_n & & xI_n + Z & \\ & & R^T.N & R^T.P.R \end{pmatrix}$$

Chapter 5

Applications

We benefit from the work carried out in the previous chapter. First, we investigate the geometry of all 4-dimensional homogeneous triples $\mathcal{H}_m^n$ that are in the scope of Definition 4.1. According to $\dim \mathfrak{m} = 2 + n + m = 4$, this concerns the triples in $\mathcal{H}_2^0, \mathcal{H}_1^1$, and $\mathcal{H}_0^2$. In particular, we derive the constraints on the parameters of the commutator to yield $R_\mathrm{o} \neq 0$, and $\mathrm{Ric}_\mathrm{o} = 0$.

Lemma 4.13 enables us to classify the triples in $\mathcal{H}_m^n$. In Section 5.2, we carry out several classifications in low dimensions. Thereby, we confirm the classification of homogeneous triples in $\mathcal{H}_0^1$, which was initially stated in [DK95].

5.1 Four-dimensional Lorentzian Ricci-flat homogeneous triples

We investigate the geometry of the 4-dimensional homogeneous triples in $\mathcal{H}_2^0, \mathcal{H}_1^1$, and $\mathcal{H}_0^2$.

Discussion 5.1. $\boxed{\mathcal{H}_2^0}$ Let $\mathfrak{n}$ be the 3-dimensional Lie algebra with basis $\langle h, r_1, r_2 \rangle$ and commutator tensor determined by

$$[,]_\mathfrak{n} = \begin{array}{c|ccc} & h & r_1 & r_2 \\ \hline h & 0 & 0 & 0 \\ r_1 & 0 & 0 & -yh \\ r_2 & 0 & yh & 0 \end{array} \qquad \text{for } y \in \mathbb{R},$$

or equivalently,

$$[u_\mathfrak{n}, v_\mathfrak{n}] = \begin{pmatrix} v_\mathbf{h} \\ v_\mathbf{r} \end{pmatrix}^T . \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} . \begin{pmatrix} u_\mathbf{h} \\ u_\mathbf{r} \end{pmatrix} h, \qquad \text{for} \quad Y = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}$$

The Lie algebra of a homogeneous triple $(\mathfrak{g}, \mathfrak{h}, L_4) \in \mathcal{H}_2^0$ is $\mathfrak{g} = \mathfrak{n} \rtimes_\delta \mathbb{R}$. The isotropy is $\mathfrak{h} = \{0\}$. We consider derivations $\delta : \mathfrak{n} \to \mathfrak{n}$ of the following form

$$\delta = \begin{pmatrix} x & g_1 & g_2 \\ 0 & p_{1,1} & p_{1,2} \\ 0 & p_{2,1} & p_{2,2} \end{pmatrix} \qquad \text{with coefficients } x, g_i, p_{i,j} \in \mathbb{R},$$

or equivalently,

$$\delta = \begin{pmatrix} x & g^T \\ 0 & P \end{pmatrix}, \qquad \text{where} \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad P = \begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{pmatrix}.$$

As derived in the previous chapter, the Jacobi identity requires $xY = P^T.Y + Y.P$. Explicitly,

$$\begin{pmatrix} 0 & yx \\ -yx & 0 \end{pmatrix} = \begin{pmatrix} -yp_{2,1} & yp_{1,1} \\ -yp_{2,2} & yp_{1,2} \end{pmatrix} + \begin{pmatrix} yp_{2,1} & yp_{2,2} \\ -yp_{1,1} & -yp_{1,2} \end{pmatrix} = \begin{pmatrix} 0 & y(p_{1,1} + p_{2,2}) \\ -y(p_{1,1} + p_{2,2}) & 0 \end{pmatrix}.$$

The matrix equation reduces to the implication $\boxed{y \neq 0 \Rightarrow x = p_{1,1} + p_{2,2}}$. Keeping this relation in mind, the commutator tensor on $\mathfrak{g}$ compiles as

$$
\mathrm{ad} = \begin{array}{r|cccc}
 & h & r_1 & r_2 & z \\
\hline
h & 0 & 0 & 0 & -xh \\
r_1 & 0 & 0 & -yh & -g_1 h - p_{1,1} r_1 - p_{2,1} r_2 \\
r_2 & 0 & yh & 0 & -g_2 h - p_{1,2} r_1 - p_{2,2} r_2 \\
z & xh & g_1 h + p_{1,1} r_1 + p_{2,1} r_2 & g_2 h + p_{1,2} r_1 + p_{2,2} r_2 & 0
\end{array}
$$

The algebra with $y = 1$, $p_{1,2} = 1$, $p_{2,1} = -1$ and all other coefficients zero is the *oscillator* algebra.

The scalar product L_4 is with respect to the basis $\mathfrak{g} = \mathfrak{m} = \langle h, r_1, r_2, z \rangle$ and induces the geometry of the homogeneous triple. We apply the results of the previous chapter to obtain the geometric tensors $\nu, \Lambda, R_\mathrm{o}, \mathrm{Ric}_\mathrm{o}$.

Adapting Lemma 4.4 to our situation gives the $(1,2)$-tensor $\nu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as

$$
\nu(u_\mathfrak{n}, v_\mathfrak{n}) = \begin{pmatrix} v_\mathbf{h} \\ v_\mathbf{r} \end{pmatrix}^T \cdot \begin{pmatrix} 0 & 0 \\ 0 & (P + P^T)/2 \end{pmatrix} \cdot \begin{pmatrix} u_\mathbf{h} \\ u_\mathbf{r} \end{pmatrix} h
$$

$$
\nu(u, z) = \begin{pmatrix} x/2 & g^T/2 & 0 \\ 0 & -(Y + P^T)/2 & -g \\ 0 & 0 & -x \end{pmatrix} \cdot \begin{pmatrix} u_\mathbf{h} \\ u_\mathbf{r} \\ u_\mathbf{z} \end{pmatrix}
$$

for all $u, v \in \mathfrak{g}$. To avoid fractions, we summarize these relations as

$$
2\nu = \begin{array}{r|cccc}
 & h & r_1 & r_2 & z \\
\hline
h & 0 & 0 & 0 & xh \\
r_1 & 0 & 2p_{1,1} h & (p_{1,2} + p_{2,1})\, h & g_1 h - p_{1,1} r_1 + (y - p_{1,2})\, r_2 \\
r_2 & 0 & (p_{1,2} + p_{2,1})\, h & 2p_{2,2} h & g_2 h - (y + p_{2,1})\, r_1 - p_{2,2} r_2 \\
z & xh & g_1 h - p_{1,1} r_1 + (y - p_{1,2})\, r_2 & g_2 h - (y + p_{2,1})\, r_1 - p_{2,2} r_2 & -2xz - 2g_1 r_1 - 2g_2 r_2
\end{array}
$$

Applying Lemma 4.6 yields the Levi-Civita connection $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ as

$$
\Lambda(u_\mathfrak{n}) = \mathcal{P}\left(\begin{pmatrix} 0 & (P + P^T + Y)/2 \end{pmatrix} . u_\mathfrak{n} \right) \qquad \text{for all } u \in \mathfrak{g}, \text{ and}
$$

$$
\Lambda(z) = \begin{pmatrix} x & g^T & 0 \\ 0 & (P - Y - P^T)/2 & -g \\ 0 & 0 & -x \end{pmatrix}.
$$

The matrices sum to

$$
P + P^T + Y = \begin{pmatrix} 2p_{1,1} & p_{1,2} + p_{2,1} + y \\ p_{1,2} + p_{2,1} - y & 2p_{2,2} \end{pmatrix}
\qquad
P - Y - P^T = \begin{pmatrix} 0 & p_{1,2} - y - p_{2,1} \\ p_{2,1} + y - p_{2,1} & 0 \end{pmatrix}
$$

Definition 4.5 declares $\mathcal{P}$, the purpose of which is to abbreviate matrices of special form. Explicitly, the Levi-Civita connection on basis elements of $\mathfrak{n}$ is $\Lambda(h) = 0$,

$$2\Lambda(r_1) = \begin{pmatrix} 0 & 2p_{1,1} & p_{1,2}+p_{2,1}-y & 0 \\ 0 & 0 & 0 & -2p_{1,1} \\ 0 & 0 & 0 & y-p_{1,2}-p_{2,1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$2\Lambda(r_2) = \begin{pmatrix} 0 & p_{1,2}+p_{2,1}+y & 2p_{2,2} & 0 \\ 0 & 0 & 0 & -y-p_{1,2}-p_{2,1} \\ 0 & 0 & 0 & -2p_{2,2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

According to Corollary 4.11, the non-zero evaluations of the curvatures $R_{\mathrm{o}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and $\mathrm{Ric}_{\mathrm{o}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ are determined by

$$\begin{aligned} R_{\mathrm{o}}(u,z) &= +\frac{1}{4}\mathcal{P}\left(\left(2(P+P^T).(P-xI_m) - (P-Y-P^T).(Y+P+P^T) - 2P^T.Y\right).u_{\mathbf{r}}\right) \\ \mathrm{Ric}_{\mathrm{o}}(z,z) &= -\frac{1}{4}\,\mathrm{tr}\,\left(2(P+P^T).(P-xI_m) + Y.Y\right) \end{aligned}$$

for all $u \in \mathfrak{g}$. A straightforward computation yields

$$\begin{aligned} R_{\mathrm{o}}(u,z) &= \frac{1}{4}\mathcal{P}\left(\begin{pmatrix} c_1+c_3 & c_2 \\ c_2 & c_1-c_3 \end{pmatrix}.u_{\mathbf{r}}\right) \\ \mathrm{Ric}_{\mathrm{o}}(z,z) &= -c_1/2 \end{aligned}$$

with coefficients c_1, c_2, c_3 as

$$\begin{cases} c_1 = -y^2 + 2p_{1,1}(p_{1,1}-x) + 2p_{2,2}\,(p_{2,2}-x) + (p_{1,2}+p_{2,1})^2 \\ c_2 = 2p_{1,1}\,(2p_{1,2}-y) - 2x\,(p_{1,2}+p_{2,1}) + 2p_{2,2}\,(y+2p_{2,1}) \\ c_3 = 2\,(p_{1,1}(p_{1,1}-x) + p_{2,2}(x-p_{2,2}) + (y-p_{1,2}+p_{2,1})\,(p_{1,2}+p_{2,1}))\,. \end{cases}$$

For convenience, we expand the curvature endomorphisms $R_{\mathrm{o}}(u,z) = \frac{1}{4}\mathcal{P}\ldots$ on basis elements

$$R_{\mathrm{o}}(r_1,z) = \frac{1}{4}\begin{pmatrix} 0 & c_1+c_3 & c_2 & 0 \\ 0 & 0 & 0 & -c_1-c_3 \\ 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{\mathrm{o}}(r_2,z) = \frac{1}{4}\begin{pmatrix} 0 & c_2 & c_1-c_3 & 0 \\ 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & c_3-c_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The curvature identities (1.6) imply $R_{\mathrm{o}}(z,r_1) = -R_{\mathrm{o}}(r_1,z)$, and $R_{\mathrm{o}}(z,r_2) = -R_{\mathrm{o}}(r_2,z)$. All other combinations of basis elements $u, v \in \{h, r_1, r_2, z\}$ yield $R_{\mathrm{o}}(u,v) = 0$.

However, if $\boxed{y \neq 0}$ we have to substitute $x = p_{1,1} + p_{2,2}$ due to the Jacobi identity. Then, the entries of the curvature tensor simplify to

$$\begin{cases} c_1 = -y^2 + (p_{1,2}+p_{2,1})^2 - 4p_{1,1}p_{2,2} \\ c_2 = 2\,(y-p_{1,2}+p_{2,1})\,(p_{2,2}-p_{1,1}) \\ c_3 = 2\,(y-p_{1,2}+p_{2,1})\,(p_{1,2}+p_{2,1})\,. \end{cases}$$

Otherwise $\boxed{y=0}$, so that

$$\begin{cases} c_1 = 2p_{1,1}(p_{1,1}-x) + 2p_{2,2}\left(p_{2,2}-x\right) + \left(p_{1,2}+p_{2,1}\right)^2 \\ c_2 = 2\left(2p_{1,1}-x\right)p_{1,2} + 2\left(2p_{2,2}-x\right)p_{2,1} \\ c_3 = 2\left(p_{1,1}(p_{1,1}-x) + p_{2,2}(x-p_{2,2}) - p_{1,2}^2 + p_{2,1}^2\right). \end{cases}$$

Relative to the basis $\mathfrak{m} = \langle h, q_1, r_1, z\rangle$, $\mathrm{Ric}_{\mathrm{o}}$ is the following bilinear form

$$\mathrm{Ric}_{\mathrm{o}}(u,v) = v^T. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_1/2 \end{pmatrix} .u \quad \text{for all } u,v\in\mathfrak{m}.$$

In any case, the equation $\mathrm{Ric}_{\mathrm{o}} = 0 \Leftrightarrow \mathrm{Ric}_{\mathrm{o}}(z,z) \Leftrightarrow c_1 = 0$ reduces to a quadratic homogeneous polynomial in the coefficients of the commutator. $\diamond$

We conjecture, that the previous discussion covers all 4-dimensional Lie algebras with scalar product of index 1, and curvature $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$.

Discussion 5.2. $\boxed{\mathcal{H}_1^1}$ The Lie algebra of a homogeneous triple in $\mathcal{H}_1^1$ is $\mathfrak{g} = \mathfrak{h}\mathfrak{e}_1 \times \mathbb{R} \rtimes_\delta \mathbb{R}$ with basis $\langle p_1, h, q_1, r_1, z\rangle$. The distinguished Lie subalgebra is $\mathfrak{h} = \langle p_1\rangle$. The derivation δ on $\mathfrak{n} = \mathfrak{h}\mathfrak{e}_1 \times \mathbb{R}$ is of the form

$$\delta = \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & x & 0 & g \\ 1 & 0 & x & 0 \\ 0 & 0 & n & p \end{pmatrix}, \qquad n,g,x,p,s\in\mathbb{R}. \tag{5.1}$$

The Lorentzian scalar product is L_4 with respect to $\mathfrak{m} = \langle h, q_1, r_1, z\rangle$. The scalar product induces the following geometric tensors. According to Lemma 4.8, the non-zero evaluations of the Riemannian curvature $R_{\mathrm{o}} : \mathfrak{m}\times\mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ are determined by

$$R_{\mathrm{o}}(u,z) = \frac{1}{4}\mathcal{P}\left(\begin{pmatrix} c_{\mathbf{q},\mathbf{q}} & c_{\mathbf{q},\mathbf{r}} \\ c_{\mathbf{r},\mathbf{q}} & c_{\mathbf{r},\mathbf{r}} \end{pmatrix} . \begin{pmatrix} u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix}\right) \qquad \text{for all } u\in\mathfrak{m},$$

where

$$\begin{aligned} c_{\mathbf{q},\mathbf{q}} &= 4s + 3nn && = 4s + 3n^2 \\ c_{\mathbf{r},\mathbf{r}} &= 2(p+p)(p-x) + (p-p)(p+p) - nn && = 4p(p-x) - n^2 \\ c_{\mathbf{q},\mathbf{r}} &= n(3p+p) - 2xn && = 2n(2p-x) \\ c_{\mathbf{r},\mathbf{q}} &= c_{\mathbf{q},\mathbf{r}} && = 2n(2p-x). \end{aligned}$$

Therefore, the essential instances of R_{o} are

$$R_{\mathrm{o}}(q_1, z) = \frac{1}{4}\begin{pmatrix} 0 & c_{\mathbf{q},\mathbf{q}} & c_{\mathbf{r},\mathbf{q}} & 0 \\ 0 & 0 & 0 & -c_{\mathbf{q},\mathbf{q}} \\ 0 & 0 & 0 & -c_{\mathbf{r},\mathbf{q}} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad R_{\mathrm{o}}(r_1, z) = \frac{1}{4}\begin{pmatrix} 0 & c_{\mathbf{q},\mathbf{r}} & c_{\mathbf{r},\mathbf{r}} & 0 \\ 0 & 0 & 0 & -c_{\mathbf{q},\mathbf{r}} \\ 0 & 0 & 0 & -c_{\mathbf{r},\mathbf{r}} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 4.9 provides the Ricci tensor on $\mathfrak{m}$. The non-zero evaluations of $\mathrm{Ric}_{\mathrm{o}} : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ are determined by

$$\mathrm{Ric}_{\mathrm{o}}(z, z) = -\frac{1}{4}\left(4s + 3nn + 2(p+p)(p-x) - nn\right) = -s + p(x-p) - n^2/2. \tag{5.2}$$

Relative to the basis $\mathfrak{m} = \langle h, q_1, r_1, z \rangle$, $\mathrm{Ric}_{\mathrm{o}}$ is the following bilinear form

$$\mathrm{Ric}_{\mathrm{o}}(u, v) = v^T.\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s + p(x-p) - n^2/2 \end{pmatrix}.u \quad \text{for all } u, v \in \mathfrak{m}.$$

Thus, $\mathrm{Ric}_{\mathrm{o}} = 0 \Leftrightarrow \boxed{s = p(x-p) - n^2/2}$. Substituting the value for s leaves the Riemannian curvature R_{o} as

$$R_{\mathrm{o}}(u, z) = \frac{1}{4}\mathcal{P}\left(\begin{pmatrix} 4p(x-p) + n^2 & 2n(2p-x) \\ 2n(2p-x) & 4p(p-x) - n^2 \end{pmatrix}.\begin{pmatrix} u_{\mathbf{q}} \\ u_{\mathbf{r}} \end{pmatrix}\right) \quad \text{for all } u \in \mathfrak{m}. \tag{5.3}$$

We investigate when the homogeneous triple is moreover Riemannian-flat. From the expression (5.3) we immediately obtain

$$R_{\mathrm{o}} = 0 \quad \Leftrightarrow \quad 2n(2p-x) = 0 \ \wedge \ n^2 + 4p(x-p) = 0.$$

Assume $n \neq 0$. Then, $R_{\mathrm{o}} = 0$ requires $x = 2p$, but the second equation $n^2 = -4p(x-p) = -4p^2$ has no real solutions for p. Hence, $R_{\mathrm{o}} = 0$ requires $n = 0$. But then $p = 0 \ \vee \ x = p$ is necessary and sufficient. Thus, we simplify the equivalence to

$$R_{\mathrm{o}} = 0 \quad \Leftrightarrow \quad n = 0 \ \wedge \ (p = 0 \ \vee \ x = p).$$

$\diamond$

Example 5.3. In [Ko01] p.69 and p.72, the homogeneous pairs with index $1.4^1.\mathbf{12}$ and index $1.4^1.\mathbf{23}$ are defined as $(\mathfrak{g}, \mathfrak{h})$ with Lie algebra $\mathfrak{g} = \mathfrak{heis}_1 \times \mathbb{R} \rtimes_{\delta_i} \mathbb{R}$ and subalgebra $\mathfrak{h} = \langle p_1 \rangle$ for $i = 12, 23$. The derivations δ_{12} and δ_{23} are stated below.

$$\delta_{12} = \begin{pmatrix} 0 & 0 & r & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \breve{\delta}_{12} = \begin{pmatrix} 0 & 0 & \frac{r}{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} & 0 & -\frac{1}{\sqrt{b}} \\ 1 & 0 & \frac{1}{\sqrt{a}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{a}} \end{pmatrix}$$

$$\delta_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \breve{\delta}_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{b}} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

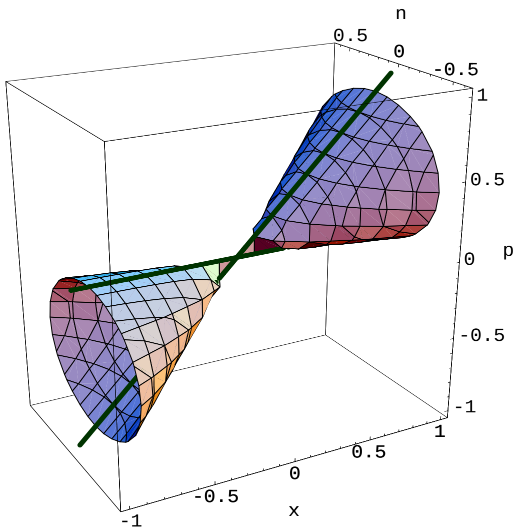

Figure 5.1 : The illustration concerns the triples in $\mathcal{H}_1^1$ as presented in Discussion 5.2. We set $s = p(x-p) - n^2/2$ for $\mathrm{Ric}_o = 0$. Then, R_o depends only on the coefficients $n, x, p \in \mathbb{R}$. The two lines represent $n = 0 \wedge (p = 0 \vee x = p)$, which is equivalent to $R_o = 0$. The surface visualizes the kernel of the bilinear form $s = p(x-p) - n^2/2 = 0$, which corresponds to $[\mathfrak{h}, \mathfrak{m}] = \{0\}$.

As part of the solution to the Einstein-Maxwell equation, the ρ-invariant Lorentzian scalar product on $\mathfrak{m} = \langle h, q_1, q_2, z\rangle$ is given by B in (3.5) with coefficients $a, b > 0$. Lemma 3.10 proves that the homogeneous triples $(\mathfrak{g}, \mathfrak{h}, B)$ are isomorphic to $\mathcal{H}_1^1(\breve{\delta}_i)$ with $\breve{\delta}_{12}, \breve{\delta}_{23}$ as stated above.

We have seen in the previous discussion, how the coefficients of the derivation relate to the curvature of the homogeneous triple. We transcribe the values in $\breve{\delta}_{12}, \breve{\delta}_{23}$ into the notation we are familiar with.

	x	p	g	n	s	$-s + p(x-p) - n^2/2$	$4s + 3n^2$	$4p(x-p) + n^2$	$2n(2p - x)$
$\breve{\delta}_{12}$	$\frac{1}{\sqrt{a}}$	$\frac{1}{\sqrt{a}}$	$-\frac{1}{\sqrt{b}}$	0	$\frac{r}{a}$	$-\frac{r}{a}$	$4\frac{r}{a}$	0	0
$\breve{\delta}_{23}$	0	0	$-\frac{1}{\sqrt{b}}$	0	0	0	0	0	0

The space $1.4.^1.23$ is Riemannian-flat. Concerning space $1.4.^1.12$, we have $\mathrm{Ric}_o = 0$ if $r = 0$, but then also $R_o = 0$. $\diamond$

Discussion 5.4. $\boxed{\mathcal{H}_0^2}$ We investigate the geometry of the homogeneous triples in $\mathcal{H}_0^2$. Let $\mathfrak{n}$ be the Lie algebra with basis $\langle p_1, p_2, h, q_1, q_2\rangle$ and commutator

$$[,]_{\mathfrak{n}} = \begin{array}{c|ccccc} & p_1 & p_2 & h & q_1 & q_2 \\ \hline p_1 & 0 & 0 & 0 & h & 0 \\ p_2 & 0 & 0 & 0 & 0 & h \\ h & 0 & 0 & 0 & 0 & 0 \\ q_1 & -h & 0 & 0 & 0 & -yh \\ q_2 & 0 & -h & 0 & yh & 0 \end{array} \qquad \text{for } y \in \mathbb{R},$$

Equivalently,

$$[u_{\mathfrak{n}}, v_{\mathfrak{n}}] = \begin{pmatrix} v_{\mathbf{p}} \\ v_{\mathbf{h}} \\ v_{\mathbf{q}} \end{pmatrix}^T \cdot \begin{pmatrix} 0 & 0 & -I_n \\ 0 & 0 & 0 \\ I_n & 0 & Z \end{pmatrix} \cdot \begin{pmatrix} u_{\mathbf{p}} \\ u_{\mathbf{h}} \\ u_{\mathbf{q}} \end{pmatrix} h \quad \text{for} \quad Z = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}.$$

The Lie algebra of a homogeneous triple $(\mathfrak{g}, \mathfrak{h}, L_4) \in \mathcal{H}_0^2$ is $\mathfrak{g} = \mathfrak{n} \rtimes_\delta \mathbb{R}$ with basis $\langle p_1, p_2, h, q_1, q_2, z \rangle$. The isotropy is $\mathfrak{h} = \langle p_1, p_2 \rangle$. We consider derivations δ on $\mathfrak{n}$ of the form

$$\delta = \begin{pmatrix} 0 & 0 & 0 & s_1 + y^2/4 & s_2 - xy/2 \\ 0 & 0 & 0 & s_2 + xy/2 & s_3 + y^2/4 \\ 0 & 0 & x & 0 & 0 \\ 1 & 0 & 0 & x & y \\ 0 & 1 & 0 & -y & x \end{pmatrix}, \qquad s_1, s_2, s_3 \in \mathbb{R},$$

or equivalently,

$$\delta = \begin{pmatrix} 0 & 0 & S - xZ/2 \\ 0 & x & 0 \\ I_2 & 0 & xI_2 + Z \end{pmatrix}, \quad \text{where} \quad S = \begin{pmatrix} s_1 + y^2/4 & s_2 \\ s_2 & s_3 + y^2/4 \end{pmatrix}.$$

With 3 degrees of freedom, S represents in fact an arbitrary symmetric matrix. The Lorentzian scalar product is L_4 with respect to $\mathfrak{m} = \langle h, q_1, q_2, z \rangle$. The scalar product induces the following geometric tensors. According to Corollary 4.12, the non-zero evaluations of $R_{\text{o}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ and $\text{Ric}_{\text{o}} : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ are determined by

$$R_{\text{o}}(u, z) = +\mathcal{P}\left((S + Z.Z/4) . u_{\mathbf{q}}\right) = \mathcal{P}\left(\begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} . u_{\mathbf{q}}\right)$$

$$\text{Ric}_{\text{o}}(z, z) = -\operatorname{tr}\,(S + Z.Z/4) = -s_1 - s_3$$

for all $u \in \mathfrak{m}$. We observe $\text{Ric}_{\text{o}} = 0 \Leftrightarrow \boxed{s_3 = -s_1}$, whilst the parameters $x, y, s_2 \in \mathbb{R}$ are free to choose. The homogeneous triple is Riemannian-flat if $s_1, s_2, s_3 = 0$. ◇

5.2 Classifications in low dimensions

Recall the definition of the linear mappings $T^{\mathbf{h}}(\lambda), T^{\mathbf{q}}(Q), T^{\mathbf{r}}(R), T^{\mathbf{z}}(\eta)$ in Table 4.1. Throughout this section, we assume that these four types generate all isomorphisms between homogeneous triples in $\mathcal{H}_m^n$. That is, we assume the moduli space is

$$\mathcal{M}_m^n = \mathcal{H}_m^n/\text{isomorphy} = \mathcal{H}_m^n/\left\langle T^{\mathbf{h}}(\mathbb{R}^*), T^{\mathbf{q}}(\mathrm{O}(\mathbb{R}^n)), T^{\mathbf{r}}\left(\mathrm{O}(\mathbb{R}^m)\right), T^{\mathbf{z}}(\mathbb{R}^m)\right\rangle.$$

We derive $\mathcal{M}_0^0, \mathcal{M}_1^0$, and $\mathcal{M}_0^1$ below, which covers the 2- and 3-dimensional triples of type $\mathcal{H}_m^n$. For greater values n, m we restrict the classification to homogeneous triples in $\mathcal{H}_m^n$ that are moreover Ricci-flat. We define

$$\dot{\mathcal{M}}_m^n = \{\tau \in \mathcal{H}_m^n : \tau \text{ is Ricci-flat}\}/\text{isomorphy}.$$

In particular, we give parametrizations of the moduli spaces $\dot{\mathcal{M}}_1^1$, and $\dot{\mathcal{M}}_0^2$.

Example 5.5. $\boxed{\mathcal{M}_0^0}$ A homogeneous triple $(\mathfrak{g}, \mathfrak{h}, L_2) = \mathcal{H}_0^0(\delta)$ has Lie algebra $\mathfrak{g} = \mathbb{R} \rtimes_\delta \mathbb{R}$, and isotropy $\mathfrak{h} = \{0\}$. The derivation is determined by a constant $x \in \mathbb{R}$. $\delta : \mathbb{R} \to \mathbb{R}$ maps $h \mapsto xh$. In matrix form, $\delta = (x)$. We display the commutator tensor ad as

$$\mathrm{ad} = \begin{array}{c|cc} & h & z \\ \hline h & 0 & -xh \\ z & xh & 0 \end{array} . \quad \text{We define} \quad \mathrm{ad}_0 = \begin{array}{c|cc} & h & z \\ \hline h & 0 & 0 \\ z & 0 & 0 \end{array} , \quad \text{and} \quad \mathrm{ad}_1 = \begin{array}{c|cc} & h & z \\ \hline h & 0 & -h \\ z & h & 0 \end{array} .$$

If $x = 0$ the triple $(\mathfrak{g}, \mathfrak{h}, L_2)$ is $\mathcal{H}_0^0((0))$ with commutator ad_0. If $x \neq 0$ the triple $\mathcal{H}_0^0((x))$ is isomorphic to $\mathcal{H}_0^0((1))$ with commutator ad_1. According to Lemma 4.13, the isomorphism is given by $\alpha = T^{\mathbf{h}}(x) = \left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)$. Thus, the moduli space $\mathcal{M}_0^0$ is a set with two points

$$\mathcal{M}_0^0 \simeq \{\mathcal{H}_0^0(\delta) : \delta = (x) \text{ with } x \in \{0, 1\}\}.$$

Any homogeneous triple in $\mathcal{H}_0^0$ is isomorphic to $\mathcal{H}_0^0(\delta)$ with exactly one derivation from the set $\delta \in \{(0), (1)\}$. In the classification of low dimensional semi-Riemannian homogeneous triples given by B. Doubrov and B. Komrakov, $\mathcal{H}_0^0((0))$ corresponds to the space 1.2, while $\mathcal{H}_0^0((1))$ corresponds to 2.3, [DK95] p.5. $\diamond$

3-dimensional homogeneous triples

The following result is on certain 3-dimensional Lie algebras $\mathfrak{g}$ with Lorentzian scalar product. The classification is not covered by [DK95], since for 3-dimensional triples $(\mathfrak{g}, \mathfrak{h}, B)$, the authors restrict to isotropy with $\dim \mathfrak{h} \geq 1$.

Lemma 5.6. $\boxed{\mathcal{M}_1^0}$ Any homogeneous triple in $\mathcal{H}_1^0$ is isomorphic to $\mathcal{H}_1^0(\delta)$ with exactly one derivation from the set

$$\delta \in \left\{ \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} : \epsilon \in \{0, 1\}, g \geq 0, p \neq 1 \right\} \simeq \mathcal{M}_1^0. \quad (5.4)$$

Proof. Recall Definition 4.1. A homogeneous triple $(\mathfrak{g}, \mathfrak{h}, L_3) \in \mathcal{H}_1^0$ consists of the semi-direct product $\mathfrak{g} = \mathbb{R}^2 \rtimes_\delta \mathbb{R}$, and isotropy $\mathfrak{h} = \{0\}$. The derivation $\delta : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the matrix $\delta = \left(\begin{smallmatrix} x & g \\ 0 & p \end{smallmatrix}\right)$, for $x, p, g \in \mathbb{R}$.

If $x \neq p$, transformation of the Lie algebra by $T^{\mathbf{z}}(\frac{g}{p-x}) : \mathfrak{g} \to \mathfrak{g}$ shows that $\mathcal{H}_1^0(\delta)$ and $\mathcal{H}_1^0\left(\left(\begin{smallmatrix} x & 0 \\ 0 & p \end{smallmatrix}\right)\right)$ are isomorphic. In this case, $g = 0$ is not a restriction. If $x = p$, we may still assume $g \geq 0$, since $\mathcal{H}_1^0\left(\left(\begin{smallmatrix} x & g \\ 0 & x \end{smallmatrix}\right)\right)$ and $\mathcal{H}_1^0\left(\left(\begin{smallmatrix} x & -g \\ 0 & x \end{smallmatrix}\right)\right)$ are isomorphic by $T^{\mathbf{r}}((-1))$. As in the previous example, we apply the mapping $T^{\mathbf{h}}$ to scale the first non-zero element in the sequence x, p, g to 1. Overall, we obtain the parametrization of derivations in (5.4), which is one-to-one with all pairwise non-isomorphic homogeneous triples in $\mathcal{H}_1^0$. $\square$

Another set of derivations that parametrizes $\mathcal{M}_1^0$ is

$$\delta \in \left\{ \begin{pmatrix} x & \epsilon \\ 0 & x \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} : \epsilon \in \{0, 1\}, x \geq 0, p \neq 1 \right\} \simeq \mathcal{M}_1^0.$$

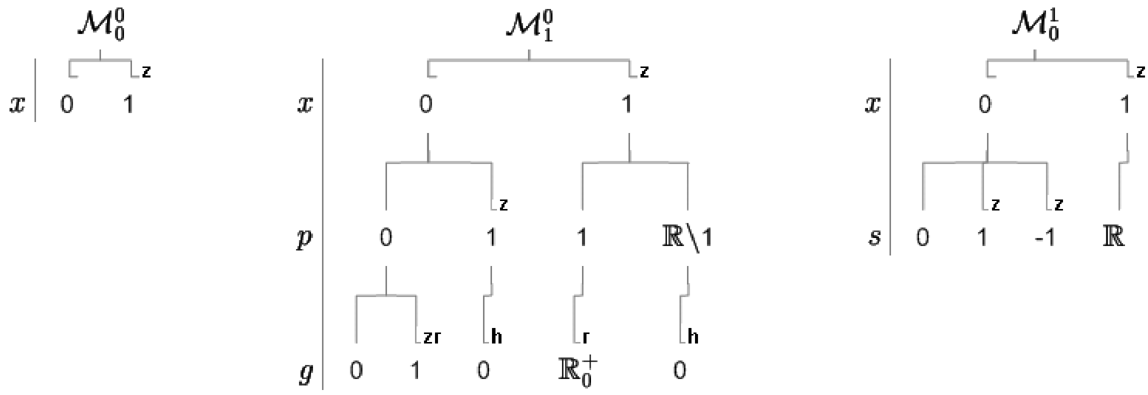

Figure 5.2 : The tree-like structures minute the reduction process of the coefficients x, p, g, s of the derivations. The bold letters $\mathbf{h}, \mathbf{r}, \mathbf{z}$ abbreviate the isomorphisms $T^{\mathbf{h}}, T^{\mathbf{r}}, T^{\mathbf{z}}$.

Next, we confirm the classification of 3-dimensional Lorentzian homogeneous triples of the form $(\mathfrak{hr}_1 \rtimes_\delta \mathbb{R}, \langle p_1 \rangle, B)$ that is stated in [DK95]. Their results are reproduced in Example 2.2. We proceed in the following way: In the following lemma, we derive our own classification. The proof shows that without loss of generality $B = L_3$. Thus the classification reduces to determine $\mathcal{M}_0^1$. We conclude by matching both classifications.

Lemma 5.7. $\boxed{\mathcal{M}_0^1}$ Any Lorentzian homogeneous triple of the form $(\mathfrak{hr}_1 \rtimes_\delta \mathbb{R}, \langle p_1 \rangle, B)$ is isomorphic to $\mathcal{H}_0^1(\tilde{\delta})$ with exactly one derivation from the set

$$\tilde{\delta} \in \left\{ \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & s \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} : \epsilon \in \{0, -1, 1\}, s \in \mathbb{R} \right\} \simeq \mathcal{M}_0^1. \tag{5.5}$$

Proof. In Section 4.1, we have derived the degrees of freedom of a linear mapping $\delta : \mathfrak{hr}_1 \to \mathfrak{hr}_1$ that is also a derivation. We may assume

$$\delta = \begin{pmatrix} 0 & 0 & s \\ 0 & x & 0 \\ 1 & 0 & x \end{pmatrix} \quad \text{for } x, s \in \mathbb{R}. \quad \text{A priori} \quad B = \begin{pmatrix} 0 & 0 & a \\ 0 & a & 0 \\ a & 0 & b \end{pmatrix} \quad \text{with } a, b \in \mathbb{R},\ a > 0. \tag{5.6}$$

To see that B covers all ρ-invariant Lorentzian scalar products on $\mathfrak{m}$, we yield the isotropy representation $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m})$. The commutator relations

$$\left. \begin{aligned} [p_1, h] &= 0 \\ [p_1, q_1] &= h \\ [p_1, z] &= -\delta(p_1) = -q_1 \end{aligned} \right\} \quad \text{define } \rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m}) \text{ as} \quad \rho(p_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The ρ-invariance condition $\rho(p_1)^T.A + A.\rho(p_1) = 0$ of a $(0,2)$-tensor $A : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ on $\mathfrak{m}$ shows that there are 2 degrees of freedom in A, if A is moreover symmetric. The general setup results in

$$A = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_2 & c_4 & c_5 \\ c_3 & c_5 & c_6 \end{pmatrix} \quad \Bigg| \quad \rho(p_1)^T.A = \begin{pmatrix} 0 & 0 & 0 \\ c_1 & c_2 & c_3 \\ -c_2 & -c_4 & -c_5 \end{pmatrix} \quad A.\rho(p_1) = \begin{pmatrix} 0 & c_1 & -c_2 \\ 0 & c_2 & -c_4 \\ 0 & c_3 & -c_5 \end{pmatrix} \quad \Bigg| \quad \rho(p_1)^T.A + A.\rho(p_1) = \begin{pmatrix} 0 & c_1 & -c_2 \\ c_1 & 2c_2 & c_3 - c_4 \\ -c_2 & c_3 - c_4 & -2c_5 \end{pmatrix}$$

for coefficients $c_i \in \mathbb{R}$. The ρ-invariance implies $c_1, c_2, c_5 = 0$, and $c_3 = c_4$. Taking $a = c_3$ and $b = c_6$ gives the scalar product B in (5.6). The determinant of B is $\det B = -a^3$, thus B is Lorentzian if $a > 0$.

In the spirit of Remark 1.9, we are looking for a Lie algebra isomorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ such that the diagram

$$\begin{array}{cccccc} L_3 : & \mathfrak{m} & \times & \mathfrak{m} & \to & \mathbb{R} \\ & \downarrow \alpha_{\mathfrak{m}} & & \downarrow \alpha_{\mathfrak{m}} & & \uparrow \mathrm{Id} \\ B : & \mathfrak{m} & \times & \mathfrak{m} & \to & \mathbb{R} \end{array}$$

commutes. We may chose $\alpha : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\alpha = \left(\begin{array}{c|c} \frac{1}{\sqrt{a}} & 0 \\ \hline 0 & \alpha_{\mathfrak{m}} \end{array}\right) \quad \text{with} \quad \alpha_{\mathfrak{m}} = \begin{pmatrix} \frac{1}{a} & 0 & -\frac{b}{2a} \\ 0 & \frac{1}{\sqrt{a}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In matrix notation, we easily confirm $L_3 = \alpha_{\mathfrak{m}}^T.B.\alpha_{\mathfrak{m}}$. A straightforward computation alike the proof of Lemma 3.10 shows further, that the commutator of $\mathfrak{g}$

$$\mathrm{ad} = \begin{array}{c|cccc} & p_1 & h & q_1 & z \\ \hline p_1 & 0 & 0 & h & -q_1 \\ h & 0 & 0 & 0 & -xh \\ q_1 & -h & 0 & 0 & -sp_1 - xq_1 \\ z & q_1 & xh & sp_1 + xq_1 & 0 \end{array}, \qquad x, s \in \mathbb{R}$$

is stable under the transformation $\alpha : \mathfrak{g} \to \mathfrak{g}$.

So far, we have reasoned that it is not a restriction to assume $B = L_3$. Consequently, the classification of Lorentzian homogeneous triples of the form $(\mathfrak{he}_1 \rtimes_\delta \mathbb{R}, \langle p_1 \rangle, B)$ is one-to-one with $\mathcal{M}_0^1$, which we are to obtain in the sequel.

According to Lemma 4.13, $\mathcal{H}_0^1(\delta)$ is isomorphic to $\mathcal{H}_0^1(\bar{\delta})$ with derivation as

$$\bar{\delta} = \begin{pmatrix} 0 & 0 & s/\lambda^2 \\ 0 & x/\lambda & 0 \\ 1 & 0 & x/\lambda \end{pmatrix}. \tag{5.7}$$

The isomorphism is given by $T^{\mathbf{h}}(\lambda)$ with $\lambda \in \mathbb{R}^*$. If $x = 0, s \neq 0$ we substitute $\lambda = \sqrt{|s|}$ in (5.7). If $x \neq 0$ we substitute $\lambda = x$. Overall, we obtain the parametrization of derivations in (5.5), which is one-to-one with all pairwise non-isomorphic homogeneous triples in $\mathcal{H}_0^1$. $\square$

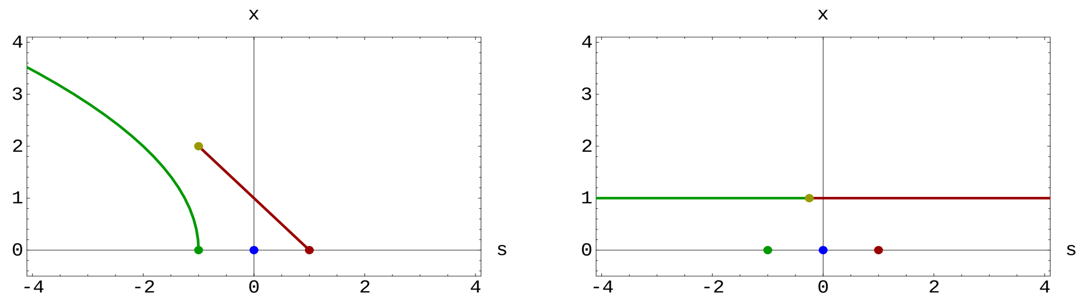

Figure 5.3 : We are interested in the classification of the homogeneous triples of type $(\mathfrak{he}_1 \rtimes_\delta \mathbb{R}, \langle p_1 \rangle, L_3)$. The graphs visualize the two different parametrizations of pairwise non-isomorphic homogeneous triples, which we compare in Corollary 5.8.

The authors of [DK95] classify all 3-dimensional semi-Riemannian homogeneous triples $(\mathfrak{g}, \mathfrak{h}, B)$ with $\dim \mathfrak{h} \geq 1$. Unfortunately, the list of triples in the paper does not come with a proof. The 3-dimensional Lorentzian homogeneous triples with Lie algebra $\mathfrak{g} = \mathfrak{he}_1 \rtimes_\delta \mathbb{R}$ and isotropy $\mathfrak{h} = \langle p_1 \rangle$ is a small excerpt of the list, which we have reproduced in Example 2.2.

Corollary 5.8. There is a one-to-one correspondence between the Lorentzian homogeneous triples $\mathcal{H}_0^1(\delta)$ listed in [DK95] and the triples $\mathcal{H}_0^1(\tilde{\delta})$ with derivation $\tilde{\delta}$ from (5.5).

Proof. We have two different sets of derivations at hand, which – provided correct – characterize all pairwise non-isomorphic Lorentzian homogeneous triples of the form $(\mathfrak{he}_1 \rtimes_\delta \mathbb{R}, \langle p_1 \rangle, B)$. Consequently, we are looking for a bijection, which couples the derivations of both sets. Applying the Lie algebra isomorphism $T^{\mathbf{h}}(\lambda)$ for appropriate values $\lambda \in \mathbb{R}^*$ does the job. The matching process is documented in Table 5.1. $\square$

4-dimensional homogeneous triples

Subsequently, we restrict the classification to homogeneous triples in $\mathcal{H}_m^n$ that are Ricci-flat. For that purpose, we define the moduli space

$$\dot{\mathcal{M}}_m^n = \{\tau \in \mathcal{H}_m^n : \tau \text{ is Ricci-flat}\} / \text{isomorphy}.$$

Since $\dim \mathfrak{m} = 2+n+m$, the 4-dimensional instances are $\dot{\mathcal{M}}_2^0, \dot{\mathcal{M}}_1^1, \dot{\mathcal{M}}_0^2$. We derive only the two latter. To derive $\dot{\mathcal{M}}_2^0$, there are 8 coefficients to consider.

Lemma 5.9. $\boxed{\dot{\mathcal{M}}_1^1}$ Any Ricci-flat triple in $\mathcal{H}_1^1$ is isomorphic to $\mathcal{H}_1^1(\delta)$ with exactly one derivation from the set

$$\delta \in \left\{ \begin{pmatrix} 0 & 0 & p(x-p) - n^2/2 & 0 \\ 0 & x & 0 & g \\ 1 & 0 & x & 0 \\ 0 & 0 & n & p \end{pmatrix} : (x,p,g,n) \in \bigcup \left\{ \begin{array}{l} \{(0,0,0,\varepsilon) : \varepsilon \in \{0,1\}\} \\ \{0\} \times \{0\} \times \{1\} \times \mathbb{R}_0^+ \\ \{0\} \times \{1\} \times \{0\} \times \mathbb{R}_0^+ \\ \{1\} \times \{1\} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \\ \{1\} \times \mathbb{R}\backslash 1 \times \{0\} \times \mathbb{R}_0^+ \end{array} \right\} \right\} \tag{5.8}$$

Table 5.1 : Bijective transcription of the derivations from Example 2.2 into the format of Lemma 5.7. The sets in the rightmost column are disjoint. Their union is $(\{0\}\times\{-1,0,1\})\bigcup(\{1\}\times\mathbb{R})$, which is literally the parametrization of derivations in (5.5).

Index		δ	λ	$\tilde{\delta}$	$(x,s)\in$
3.1		$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\{(0,0)\}$
3.3		$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$	2	$\begin{pmatrix} 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\{(1,\frac{-1}{4})\}$
3.4	$\alpha=-1$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\{(0,1)\}$
3.4	$-1<\alpha<1$	$\begin{pmatrix} 0 & 0 & -\alpha \\ 0 & \alpha+1 & 0 \\ 1 & 0 & \alpha+1 \end{pmatrix}$	$\alpha+1$	$\begin{pmatrix} 0 & 0 & -\frac{\alpha}{(\alpha+1)^2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\{1\}\times(\frac{-1}{4},\infty)$
3.5	$\alpha=0$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\{(0,-1)\}$
3.5	$0<\alpha$	$\begin{pmatrix} 0 & 0 & -\alpha^2-1 \\ 0 & 2\alpha & 0 \\ 1 & 0 & 2\alpha \end{pmatrix}$	2α	$\begin{pmatrix} 0 & 0 & -\frac{\alpha^2+1}{4\alpha^2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\{1\}\times(-\infty,\frac{-1}{4})$

Proof. According to Definition 4.1, $\delta:\mathfrak{he}_1\times\mathbb{R}\to\mathfrak{he}_1\times\mathbb{R}$ is of the general form

$$\delta=\begin{pmatrix} 0 & 0 & s & 0 \\ 0 & x & 0 & g \\ 1 & 0 & x & 0 \\ 0 & 0 & n & p \end{pmatrix},\qquad n,g,x,p,s\in\mathbb{R}. \tag{5.9}$$

In Discussion 5.2, we have derived the equivalence $\mathrm{Ric}_o=0\Leftrightarrow\boxed{s=p(x-p)-n^2/2}$. The orthogonal group on $\mathbb{R}$ is just $\mathrm{O}(\mathbb{R})=\{(1),(-1)\}$. The commutator tensors and the Lorentzian scalar product $B=L_4$ are stable under these transformations, except for the coefficients of the derivation δ on $\mathfrak{he}_1\times\mathbb{R}$. Consider the effect of three particular transformations $\alpha:\mathfrak{g}\to\mathfrak{g}$ on the

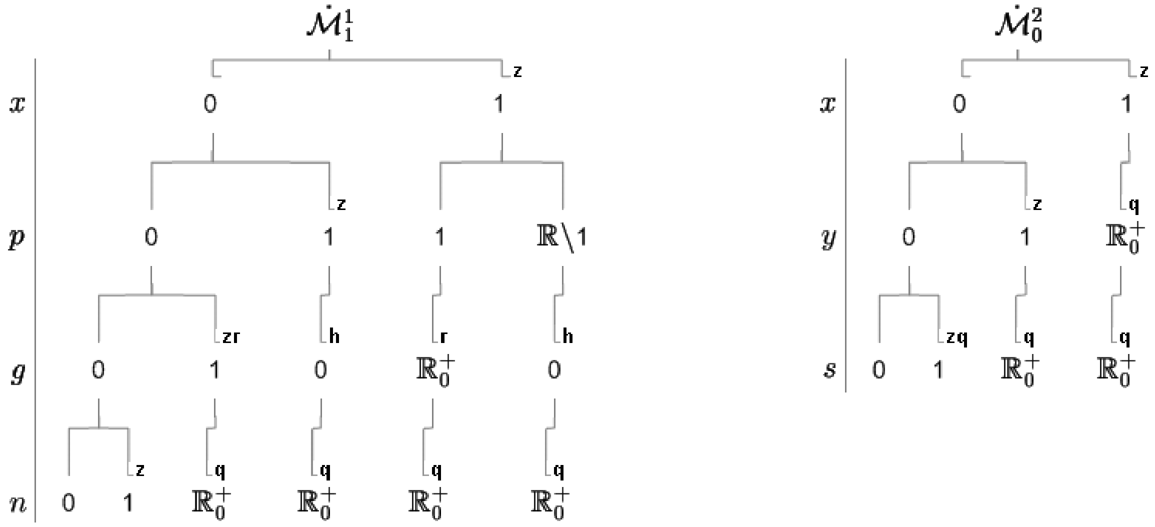

Figure 5.4 : We depict the reduction process of the coefficients x, p, g, n, y, s in the classifications that we perform in Lemma 5.9, and Lemma 5.10.

entries of the derivation:

α	$T^{\mathbf{q}}((-1)).T^{\mathbf{h}}(-1)$	$T^{\mathbf{q}}((-1)).T^{\mathbf{r}}((-1))$	$T^{\mathbf{q}}((-1))$
$\mathrm{diag}(\alpha)$	$(1,-1,-1,1,-1)$	$(-1,1,-1,-1,1)$	$(-1,1,-1,1,1)$
$\breve{\delta}$	$\begin{pmatrix} 0 & 0 & s & 0 \\ 0 & -x & 0 & g \\ 1 & 0 & -x & 0 \\ 0 & 0 & n & -p \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & s & 0 \\ 0 & x & 0 & -g \\ 1 & 0 & x & 0 \\ 0 & 0 & n & p \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & s & 0 \\ 0 & x & 0 & g \\ 1 & 0 & x & 0 \\ 0 & 0 & -n & p \end{pmatrix}$

Thus, to determine a set of pairwise non-isomorphic triples, we may narrow the discussion to derivations with $\boxed{x, g, n \geq 0}$ and $\boxed{x = 0 \Rightarrow p \geq 0}$. If $x \neq p$ we transform the Lie algebra by $T^{\mathbf{z}}(\frac{g}{p-x})$, which gives the rule $\boxed{x \neq p \Rightarrow g = 0}$. Finally, we apply $T^{\mathbf{h}}(\lambda) : \mathfrak{g} \to \mathfrak{g}$ with $\lambda \in \mathbb{R}^*$, so that the first non-zero element in the sequence x, p, g, n scales to 1. For instance, if $x, p = 0$, but $g > 0$, we choose the transformation $T^{\mathbf{h}}(\sqrt{g})$.

Overall, we obtain the parametrization of derivations in (5.8), which is one-to-one with all pairwise non-isomorphic homogeneous triples in $\mathcal{H}_1^1$. $\square$

Lemma 5.10. $\boxed{\dot{\mathcal{M}}_0^2}$ Any Ricci-flat triple in $\mathcal{H}_0^2$ is isomorphic to $\mathcal{H}_0^2(\delta)$ with exactly one derivation from the set

$$\delta \in \left\{ \begin{pmatrix} 0 & 0 & 0 & s + y^2/4 & -xy/2 \\ 0 & 0 & 0 & xy/2 & y^2/4 - s \\ 0 & 0 & x & 0 & 0 \\ 1 & 0 & 0 & x & y \\ 0 & 1 & 0 & -y & x \end{pmatrix} : (x, y, s) \in \bigcup \left\{ \begin{array}{c} \{(0,0,\epsilon) : \epsilon \in \{0,1\}\} \\ \{0\} \times \{1\} \times \mathbb{R}_0^+ \\ \{1\} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \end{array} \right\} \right\} \tag{5.10}$$

Proof. Generally, we consider derivations $\delta : \mathfrak{hx}_2 \to \mathfrak{hx}_2$ of the form

$$\delta = \begin{pmatrix} & & S - xZ/2 \\ & x & \\ I_2 & & xI_2 + Z \end{pmatrix}, \quad \text{where} \quad Z = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}, \quad \text{and } S \text{ symmetric.} \tag{5.11}$$

In Discussion 5.4, we derive the relation $\boxed{\operatorname{tr} S + Z^2/4 = 0}$ to yield $\mathrm{Ric}_\mathrm{o} = 0$. Since $Z^2/4 = -\mathrm{diag}(y^2, y^2)/4$, we choose S to be of the general form

$$S = \begin{pmatrix} s_1 + y^2/4 & s_2 \\ s_2 & y^2/4 - s_1 \end{pmatrix}, \quad s_1, s_2 \in \mathbb{R}. \quad \text{Let} \quad Q = \begin{pmatrix} \cos t/2 & \sin t/2 \\ -\sin t/2 & \cos t/2 \end{pmatrix}.$$

By transformation with $T^{\mathbf{q}}(Q) : \mathfrak{g} \to \mathfrak{g}$, the triple $\mathcal{H}_0^2(\delta)$ is isomorphic to $\mathcal{H}_0^2(\breve{\delta})$ with

$$\breve{\delta} = \begin{pmatrix} & & Q^T.S.Q - xZ/2 \\ & x & \\ I_2 & & xI_2 + Z \end{pmatrix}.$$

Note, $Q^T.Z.Q = Z$. The matrix product $Q^T.S.Q$ is

$$Q^T.S.Q = \begin{pmatrix} y^2/4 + s_1 \cos t - s_2 \sin t & s_1 \sin t + s_2 \cos t \\ s_1 \sin t + s_2 \cos t & y^2/4 - s_1 \cos t + s_2 \sin t \end{pmatrix}.$$

We choose $t \in [\pi, \pi)$, so that $0 = s_1 \sin t + s_2 \cos t$ and $s = s_1 \cos t - s_2 \sin t \geq 0$, and thereby eliminate one degree of freedom. Henceforth, we assume the derivation δ is of the form (5.11), where

$$S = \begin{pmatrix} s + y^2/4 & -xy/2 \\ xy/2 & y^2/4 - s \end{pmatrix}, \quad \text{and } \boxed{s \geq 0}.$$

The effects of two specific transformations $\alpha : \mathfrak{g} \to \mathfrak{g}$ on the derivation reveal, that $\boxed{x, y \geq 0}$ is not a restriction. Note, $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{O}(\mathbb{R}^2)$.

α	Id	$T^{\mathbf{q}}\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right).T^{\mathbf{h}}(-1)$	$T^{\mathbf{q}}\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$
diag	$(1,1,1,1,1,1)$	$(1,-1,-1,-1,1,-1)$	$(-1,1,1,-1,1,1)$
$\breve{\delta}$	$\begin{pmatrix} & & S - xZ/2 \\ & x & \\ I_2 & & xI_2 + Z \end{pmatrix}$	$\begin{pmatrix} & & S - (-x)Z/2 \\ & (-x) & \\ I_2 & & (-x)I_2 + Z \end{pmatrix}$	$\begin{pmatrix} & & S - x(-Z)/2 \\ & x & \\ I_2 & & xI_2 + (-Z) \end{pmatrix}$

Finally, we apply $T^{\mathbf{h}}(\lambda) : \mathfrak{g} \to \mathfrak{g}$ with $\lambda \in \mathbb{R}^*$, so that the first non-zero element in the sequence x, y, s scales to 1. Overall, we obtain the parametrization of derivations in (5.10), which is one-to-one with all pairwise non-isomorphic homogeneous triples in $\mathcal{H}_0^2$. $\square$

In [Ko01], B. Komrakov provides a list of pairwise non-isomorphic homogeneous *pairs* of dimension 4 and $\dim \mathfrak{h} \geq 1$. The list extends in almost 80 pages. The classification of 4-dimensional homogeneous *triples* would require an even more verbose exposition.

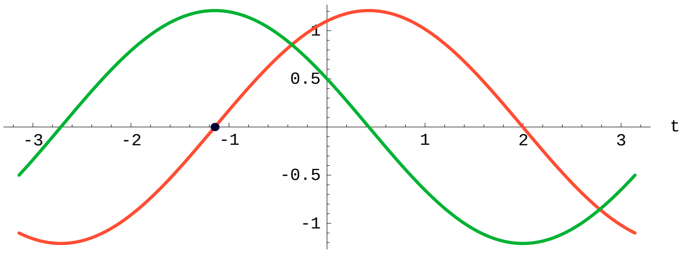

Figure 5.5 : The graphs of $s_1 \sin t + s_2 \cos t$, and $s_1 \cos t - s_2 \sin t$. The black dot indicates the value $t \in [\pi, \pi)$, so that $s_1 \sin t + s_2 \cos t = 0$, and $s_1 \cos t - s_2 \sin t \geq 0$. For the illustration, we have chosen the constants $s_1 = \frac{1}{2}, s_2 = \frac{11}{10}$.

To state the next result, we assume the classification of 4-dimensional homogeneous pairs in [Ko01] is correct. We classify all 4-dimensional homogeneous triples $(\mathfrak{g}, \mathfrak{h}, B)$ with $\dim \mathfrak{h} \geq 1$ and the geometric properties

$$\left.\begin{array}{ll} B \text{ has index } 1 & \text{the scalar product } B \text{ on } \mathfrak{m} \text{ is Lorentzian} \\ R_{\mathrm{o}} \neq 0 & \text{not Riemannian-flat} \\ \mathrm{Ric}_{\mathrm{o}} = 0 & \text{Ricci-flat} \end{array}\right\} \tag{5.12}$$

For that purpose, we define the moduli space

$$\check{\mathcal{M}}^n_m = \{\tau \in \mathcal{H}^n_m : \tau \text{ has geometry } (5.12)\} \,/\text{isomorphy}.$$

Corollary 5.11. $\boxed{\check{\mathcal{M}}^1_1, \check{\mathcal{M}}^2_0}$ Any 4-dimensional Lorentzian homogeneous triple $(\mathfrak{g}, \mathfrak{h}, B)$ with $\dim \mathfrak{h} \geq 1$ that is Ricci-flat and non-Riemannian-flat is isomorphic to either $\mathcal{H}^1_1(\delta_1)$, or $\mathcal{H}^2_0(\delta_2)$ with exactly one derivation from the set

$$\delta_1 \in \left\{ \begin{pmatrix} 0 & 0 & p(x-p) - n^2/2 & 0 \\ 0 & x & 0 & g \\ 1 & 0 & x & 0 \\ 0 & 0 & n & p \end{pmatrix} : (x,p,g,n) \in \bigcup \left\{ \begin{array}{l} \{(0,0,0,1)\} \\ \{0\} \times \{0\} \times \{1\} \times \mathbb{R}^+ \\ \{0\} \times \{1\} \times \{0\} \times \mathbb{R}^+_0 \\ \{1\} \times \{1\} \times \mathbb{R}^+_0 \times \mathbb{R}^+ \\ \{1\} \times \{0\} \times \{0\} \times \mathbb{R}^+ \\ \{1\} \times \mathbb{R}\backslash\{0,1\} \times \{0\} \times \mathbb{R}^+_0 \end{array} \right\} \right\} \simeq \check{\mathcal{M}}^1_1$$

$$\delta_2 \in \left\{ \begin{pmatrix} 0 & 0 & 0 & s + y^2/4 & -xy/2 \\ 0 & 0 & 0 & xy/2 & y^2/4 - s \\ 0 & 0 & x & 0 & 0 \\ 1 & 0 & 0 & x & y \\ 0 & 1 & 0 & -y & x \end{pmatrix} : (x,y,s) \in \bigcup \left\{ \begin{array}{l} \{(0,0,1)\} \\ \{0\} \times \{1\} \times \mathbb{R}^+ \\ \{1\} \times \mathbb{R}^+_0 \times \mathbb{R}^+ \end{array} \right\} \right\} \simeq \check{\mathcal{M}}^2_0$$

Proof. Let $(\mathfrak{g}, \mathfrak{h}, B)$ be a 4-dimensional Lorentzian homogeneous triple with curvature $R_{\mathrm{o}} \neq 0$, and $\mathrm{Ric}_{\mathrm{o}} = 0$. According to Corollary 4.3, this triple is isomorphic to an element in either $\mathcal{H}^1_1$, or $\mathcal{H}^2_0$. Thus, the classification of all such triples reduces to determine $\check{\mathcal{M}}^1_1$, and $\check{\mathcal{M}}^2_0$.

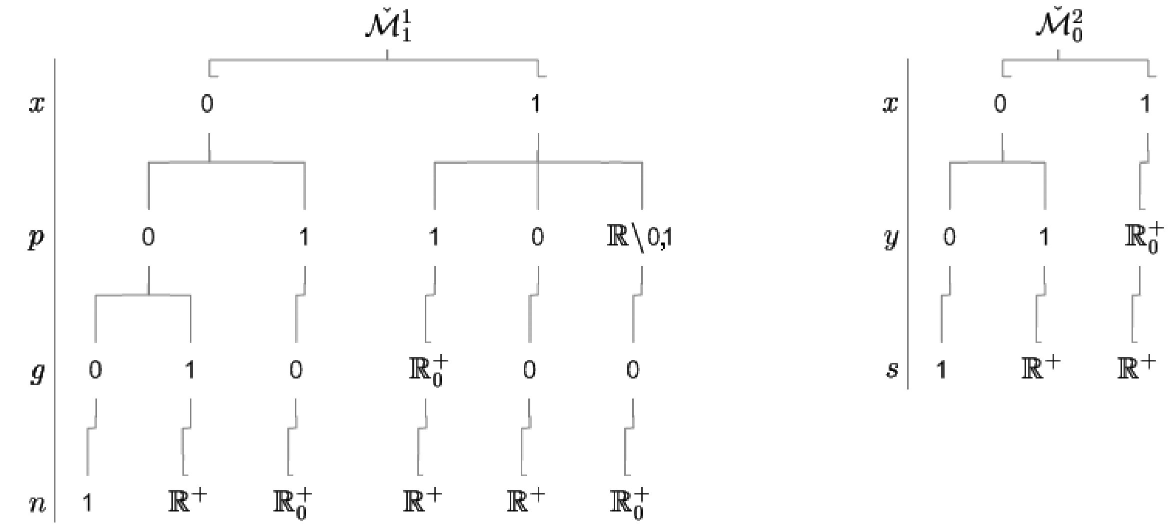

Figure 5.6 : To obtain the moduli spaces $\check{\mathcal{M}}_1^1$ and $\check{\mathcal{M}}_0^2$, we simply remove Riemannian-flat spaces from $\dot{\mathcal{M}}_1^1$ and $\dot{\mathcal{M}}_0^2$. The classification is carried out in Corollary 5.11.

The space $\dot{\mathcal{M}}_m^n$ consists of all pairwise non-isomorphic homogeneous triples in $\mathcal{H}_m^n$ that are Ricci-flat. So, $\check{\mathcal{M}}_m^n$ is merely $\dot{\mathcal{M}}_m^n$ with Riemannian-flat spaces removed.

Lemmas 5.9, 5.10 parametrize $\dot{\mathcal{M}}_1^1$, $\dot{\mathcal{M}}_0^2$ through the sets of derivations (5.8), (5.10). In terms of the coefficients of the derivation, a triple $\mathcal{H}_1^1(\delta)$ with δ as in (5.8) is Riemannian-flat if $\boxed{n = 0 \wedge (p = 0 \vee x = p)}$. This is a result of Discussion 5.2. According to Discussion 5.4, a triple $\mathcal{H}_0^2(\delta)$ with δ as in (5.10) is Riemannian-flat if $\boxed{s = 0}$.

We remove the derivations from the sets (5.8), (5.10), which satisfy these criteria. This establishes a one-to-one correspondence with $\check{\mathcal{M}}_1^1$, $\check{\mathcal{M}}_0^2$ and proves the claim. □

5.3 Future work

The thesis has obvious shortcomings: There is a fool-proof method to construct a Lie group with Lie algebra $\mathfrak{n} \rtimes_\delta \mathbb{R}$, where $\mathfrak{n}$ is a nilpotent Lie algebra with basis so that $\delta : \mathfrak{n} \to \mathfrak{n}$ is in real Jordan normal form. Applying an appropriate basis transformation, any triple $\mathcal{H}_m^n(Y, \delta)$ is of this form. However, I do not know of a universal method to explicitly contruct a homogeneous space with associated homogeneous triple $\mathcal{H}_m^n(Y, \delta)$.

In [DK95], the parametrization of homogeneous triples in $\mathcal{M}_0^1$ is closely related to the Jordan normal form of the derivation. This seems to be advantageous in order to formulate the left-action of the corresponding homogeneous space. I yield new moduli spaces, for instance $\check{\mathcal{M}}_1^1, \check{\mathcal{M}}_0^2$, in the most straight-forward way. That is, I neglect the Jordan normal form of the derivation. However, I hope that the classifications serve as a good reference for future parametrizations of these moduli spaces.

With my current state of knowledge, I am not able to report on the applications of Lorentzian Ricci-flat homogeneous spaces to relativistic physics.

Bibliography

[Ar03] A. Arvanitoyeorgos: *An Introduction to Lie Groups and the Geometry of Homogeneous Spaces*, Student Mathematical Library, Volume 22, American Mathematical Society, 2003

[AS04] A. Agrachev, Y. Sachkov: *Control Theory from the Geometric Viewpoint*, EMS, Volume 87, Springer, 2004

[Ba89] H. Baum: *Complete Riemannian Manifolds with Imaginary Killing Spinors*, Annals of Global Analysis and Geometry, 7(3) pp. 205–226, 1989

[Ba05] H. Baum: *Einführung in die Eichfeldtheorie*, Vorlesungsskript, 2005

[Be80] A. Besse: *Einstein Manifolds*, American Mathematical Society, 1980

[BG05] K. Burns, M. Gidea: *Differential Geometry and Topology*, Chapman & Hall, 2005

[Bo86] W. Boothby: *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Academic Press, 1986

[Bu85] W. Burke: *Applied Differential Geometry*, Cambridge University Press, 1995

[CW70] M. Cahen, N. Wallach: *Lorentzian Symmetric Spaces*, Bulletin of the AMS 76, 1970, pp. 585–591.

[DK95] B. Doubrov, B. Komrakov: *Low-dimensional Pseudo-Riemannian Homogeneous Spaces*, Preprint University of Oslo, No. 13, March 1995

[DK00] J. Duistermaat, J. Kolk: *Lie-Groups*, Springer, 2000

[FM05] J. Figueroa-O'Farrill, P. Meessen, S. Philip: *Homogeneity and Plane-Wave Limits*, Journal of High Energy Physics, Volume 5, 2005

[FR05] M. Fels, A. Renner: *Non-reductive Homogeneous Pseudo-Riemannian Manifolds of Dimension Four*, arXiv math, Differential Geometry/0406147v2, 2005

[Ha06] J. Hakenberg: *Differential Geometry*, `www.hakenberg.de`, 2006

[HN91] J. Hilgert, K. Neeb: *Lie-Gruppen und Lie-Algebren*, Vieweg, 1991

[KC93] B. Komrakov, A. Churyumov, B. Doubrov: *Two-dimensional Homogeneous Spaces*, Preprint Series - Matematisk Institutt, Universitetet i Oslo, 1993

[KN69] S. Kobayashi, K. Nomizu: *Foundations of Differential Geometry*, Wiley, 1969

[Ko95] B. Komrakov: *Four-dimensional Pseudo-Riemannian Homogeneous Spaces. Classification of real pairs*, Preprint University of Oslo, No. 32 June, 1995

[Ko01] B. Komrakov: *Einstein-Maxwell Equation on four-dimensional Homogeneous Spaces*, Lobachevskii Journal of Mathematics, Volume 8, pp. 33–165, 2001

[Kü03] W. Kühnel: *Differential Geometrie*, Vieweg, 2. Auflage, 2003

[Lie30] S. Lie: *Theorie der Transformationsgruppen · Erster Abschnitt*, B. G. Teubner, 1930

[Ne02] T. Neukirchner: *Pseudo-Riemannian Symmetric Spaces*, Diplomarbeit, Mathematik HU-Berlin, 2002

[Ne03] T. Neukirchner: *Solvable Pseudo-Riemannian Symmetric Spaces*, arXiv math, Differential Geometry/0301326, 2003

[ON83] B. O'Neill: *Semi-Riemannian Geometry*, Academic Press, 1983

[Re95] U. Reif: *A Unified Approach to Subdivision Algorithms near Extraordinary Points*, Computer-Aided Geometric Design 12, pp. 153–174, 1995

[Sa97] T. Sakai: *Riemannian Geometry*, Translations of Mathematical Monographs, Volume 149, American Mathematical Society, 1997

[SW04] S. Schaefer, J. Warren: *Smooth Subdivision of Tetrahedral Meshes*, Eurographics Symposium on Geometry Processing, pp. 151–158, 2004

Declaration

I hereby certify that this thesis has been composed by me and is based on my own work, unless stated otherwise. This work has not been submitted for any other degree.

Berlin, 30th September 2006
Jan Philipp Hakenberg

Theses

Let us accentuate the major contributions of this thesis.

- Suppose the classification of 4-dimensional homogeneous pairs in [Ko01] is correct. Then, Corollary 3.12 shows that any 4-dimensional Lorentzian homogeneous triple with isotropy $\dim \mathfrak{h} \geq 1$, and curvature $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$ is isomorphic to $(\mathfrak{g}, \mathfrak{h}, L_4)$ with $\mathfrak{g} = \mathfrak{he}_n \times \mathbb{R}^m \rtimes_\delta \mathbb{R}$, and $\mathfrak{h} = \langle p_1, \dots, p_n \rangle$ for either $n = 1, m = 1$, or $n = 2, m = 0$.
- We define a new class of Lorentzian homogeneous triples $\mathcal{H}_m^n$ in 4.1. The class incorporates all homogeneous triples with geometry $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$ that are known to us. We indicate the connection to previous work such as [CW70], and [FM05].
- The geometry of triples in $\mathcal{H}_m^n$ with $m \neq 0$ has not been published prior to this thesis. The set of all triples of the form $\mathcal{H}_m^n(Y, \delta)$ for fixed n, and $m \geq 2$ is not a vector space. The triples in $\mathcal{H}_m^0$ have isotropy of dimension 0. In this case, the corresponding homogeneous spaces are Lie groups with left-invariant metric.
- We present in detail the geometry of all 4-dimensional homogeneous triples that originate from our construction $\mathcal{H}_m^n$. We emphasize on the choice of parameters in Y, δ, such that the curvature satisfies $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$. In particular, we yield instances of homogeneous triples, which correspond to 4-dimensional Lie groups with left-invariant Lorentzian metric of this geometry.
- Suppose the classification of 4-dimensional homogeneous pairs in [Ko01] is correct. Then, Corollary 5.11 classifies the 4-dimensional Lorentzian homogeneous triples with isotropy $\dim \mathfrak{h} \geq 1$, and curvature $R_{\mathrm{o}} \neq 0, \mathrm{Ric}_{\mathrm{o}} = 0$.