Inverse Distance Coordinates for Scattered Sets of Points

Jan Hakenberg
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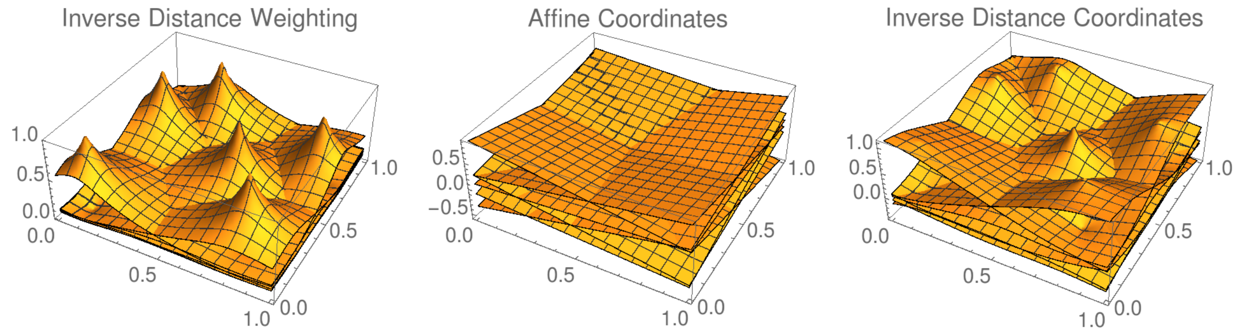

Figure: Basis functions of inverse distance weighting, affine coordinates, and inverse distance coordinates for an example set of six points in the plane. ■

Abstract: We present meshfree generalized barycentric coordinates for scattered sets of points in d-dimensional vector space. The coordinates satisfy the Lagrange property. Our derivation is based on the projection of Shepard's popular inverse distance weights to their best fit in the subspace of coordinates with linear reproduction. The notion of distance between a pair of points is sufficient for the construction of coordinates.

Keywords: meshfree generalized barycentric coordinates, Lagrange property, zwischenzug

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Introduction

Given a set of pairwise distinct points $p_1, p_2, \ldots, p_n \in \mathbb{R}^d$ with $n > d$, we construct a generalized barycentric coordinate $w \in \mathbb{R}^n$ that depends on a choice of vector norm in $\mathbb{R}^d$ and the location of evaluation $x \in \mathbb{R}^d$. By construction the coordinate w shall satisfy the following properties with respect to x

(1) $\sum_{i=1}^{n} w_i = 1$ (partition of unity)

$\sum_{i=1}^{n} p_i\, w_i = x$ (linear reproduction)

We shift the points p_i so that "x becomes the origin" using the translation $v_i := p_i - x$ for $i = 1, 2, \ldots, n$ that results in the equivalent condition

(2) $\sum_{i=1}^{n} v_i\, w_i = 0$ (linear reproduction)

At the isolated parameters $x = p_i$ for $i = 1, \ldots, n$ from the input set our construction satisfies

(3) $x = p_i \;\Rightarrow\; w_j = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$ (Lagrange property)

Related work

[1968 Donald Shepard] introduced *inverse distance weighting* for the interpolation of irregularly-spaced data. His method does not reproduce linear polynomials. Our approach projects the inverse distance weights of Shepard's method to the closest solution of (2) in the least-square sense.

[2011 Shayne Waldron] derives *affine coordinates* that satisfy (2), but do not have the Lagrange property. Waldron's approach selects the arithmetic mean $m = (1/n) \sum_{i=1}^{n} p_i$ of the set of input points as the fixed reference for the best fit to coordinates with linear reproduction. Affine coordinates require the computation of a pseudoinverse matrix only once for a given set of points. In contrast, the inverse distance coordinates presented in this article require a best fit at every parameter x and therefore involve the computation of nullspace and pseudoinverse at every evaluation.

The book [2016 Hormann, Sukumar] lists two additional generalized barycentric coordinates for scattered sets of points: *Sibson coordinates* and *Laplace coordinates*, see Sections 1.2.12 and 1.2.13 in the book. Both methods involve the construction of Voronoi cells and are at most C^1. Sibson coordinates rely on the notion of area, or volume.

Construction

We state the derivation of the new coordinates that were coined *Inverse Distance Coordinates* by Kai Hormann upon reviewing an early version of this article.

Any non-zero vector from the nullspace of the matrix $V = [v_1, \ldots, v_n]$ of dimensions $d \times n$ that may be normalized to sum up to 1 produces a solution to (1) and (2).

Denote with $N = \text{nullspace}(V)$ the matrix of dimensions $r \times n$ with row vectors that span the nullspace of V, i.e. that satisfies $V.N^T = 0$. The nullspace is non-trivial because $n > d$ implies $r > 0$.

The best fit $\tilde{w} \in \mathbb{R}^n$ in the least-square sense to *target weights* $\alpha \in \mathbb{R}^n$ that also satisfies (2) is the vector

(4) $\tilde{w} = \alpha.N^+.N$

where N^+ denotes the pseudoinverse of N with dimensions $n \times r$.

If $\sum_{i=1}^n \tilde{w}_i \neq 0$, we obtain the vector of coordinates $w \in \mathbb{R}^n$ that additionally satisfies (1) using scaling

(5) $w = \frac{1}{\sum_{i=1}^n \tilde{w}_i} \tilde{w}$

The degree of freedom is in the design of *target weights* $\alpha \in \mathbb{R}^n$ that ensure the convergence of coordinates w to satisfy (3) as x approaches a point p_i from the input set.

A straight forward choice for $\alpha \in \mathbb{R}^n$ is the vector of inverse distances between p_i and x, or equivalently the inverse norms of v_i

$$\alpha_i := 1 / \| p_i - x \| = 1 / \| v_i \| \qquad \text{for } i = 1, \ldots, n,$$

which is well-defined if $x \neq p_i$ for all $i = 1, \ldots, n$. $\| v_i \|$ denotes the Euclidean norm, i.e. 2-norm of the vector $v_i \in \mathbb{R}^d$.

As $x \to p_i$, the $n \times n$ projection matrix $M := N^+.N$ in (4) converges to have entries $M_{i,j} = M_{j,i} = \delta_{i,j}$ for $j = 1, \ldots, n$. That means, the entry α_i that tends to infinity as $v_i \to 0$ only manifests itself in the entry $\tilde{w}_i$ in (4). The normalization in (5) results in the convergence of w to the unit vector e_i thereby establishing the Lagrange property.

The construction is well-defined for any x that results in $\alpha < \infty$ and $\sum_{i=1}^n \tilde{w}_i \neq 0$. The coordinates w are C^∞ around such points x, since the construction consists of smooth operations entirely.

Remark: The design of other non-zero target weights α is subject to future work. ■

Implementation and Results

The implementations in *Mathematica* illustrate the similarities between inverse distance weighting, affine coordinates, and inverse distance coordinates:

```
Shepard[points_, x_] := Module[{Vt = # - x & /@ points, α},
  α = 1 / Norm /@ Vt;
  Normalize[α, Total]]
```

```
Waldron[points_, x_] := Module[{m = Mean[points], Vt},
  Vt = # - m & /@ points;
  (x - m).PseudoInverse[Vt] + 1 / Length[points]]
```

```
IDC[points_, x_] := Module[{Vt = # - x & /@ points, α, N},
  α = 1 / Norm /@ Vt;
  N = NullSpace[Transpose[Vt]];
  Normalize[α.PseudoInverse[N].N, Total]]
```

Example: Let $p_1 = (0.1, 0.1)$, $p_2 = (0.8, 0.2)$, $p_3 = (0.9, 0.7)$, $p_4 = (0.6, 0.5)$, $p_5 = (0.3, 0.9)$, $p_6 = (0.1, 0.7)$, and $x = (0.3, 0.4)$. The inverse distance weights evaluate to $w_{\text{IDW}} \approx (0.197, 0.132, 0.106, 0.225, 0.142, 0.197)$.

The affine coordinate is $w_{AC} \approx (0.368, 0.152, 0.003, 0.134, 0.119, 0.223)$. The inverse distance coordinate evaluates to $w_{IDC} \approx (0.350, 0.126, -0.022, 0.223, 0.089, 0.234)$.

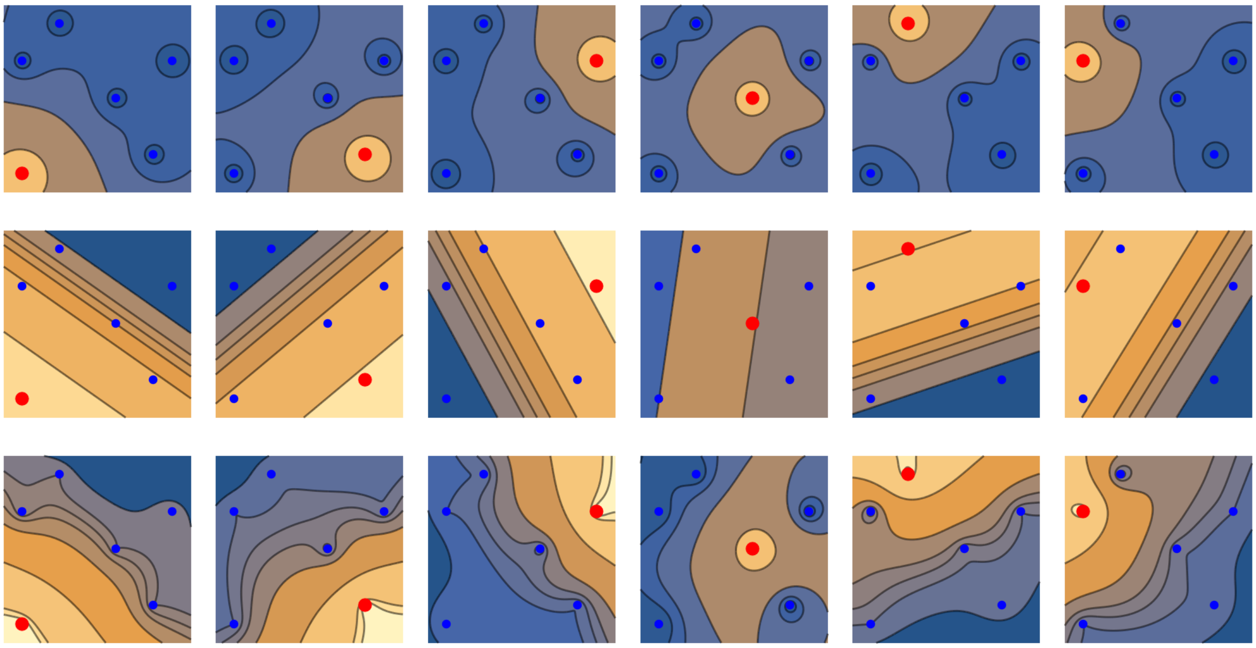

Figure: Approximate contour plots of the basis functions w_i for $i = 1, \ldots, 6$ and the point set from the example evaluated over the unit square. From top to bottom: Inverse distance weighting, affine coordinates, and inverse distance coordinates. ■

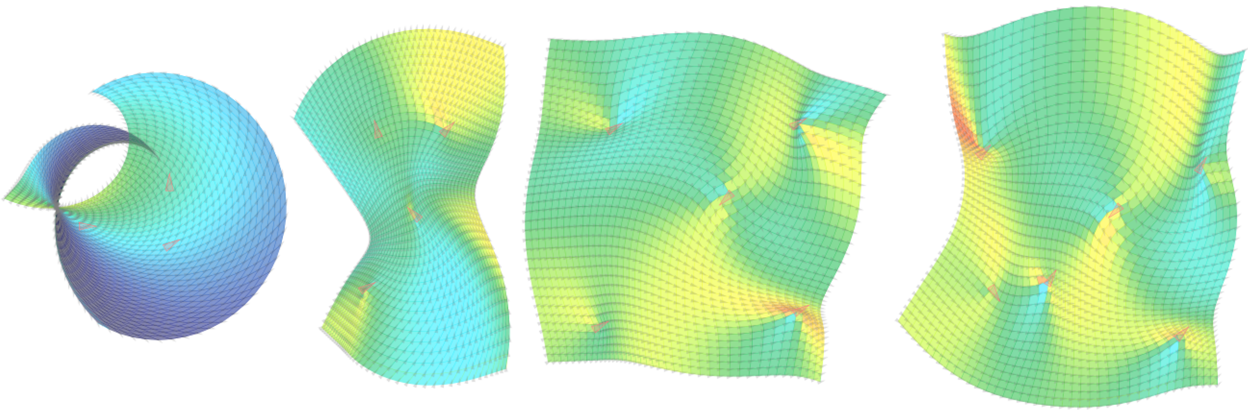

Figure: The graphics show the evaluation of inverse distance coordinates for input sets of points $p_i = (px_i, py_i)$ from the plane for $n = 3, 4, 5, 6$, respectively, that have associated values in the 3-dimensional Lie group $\overline{\text{SE}}(2)$ of the form (px_i, py_i, θ_i). The coordinates are evaluated over a rectangular domain that covers the point sets and mapped into the Lie group using the formulas derived in [2006 Arsigny, Pennec, Ayache]. Elements from $\overline{\text{SE}}(2)$ are plotted as arrowheads. ■

References

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2006 Vincent Arsigny, Xavier Pennec, Nicholas Ayache: *Bi-invariant Means in Lie Groups. Application to left-invariant Polyaffine Transformations.*
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