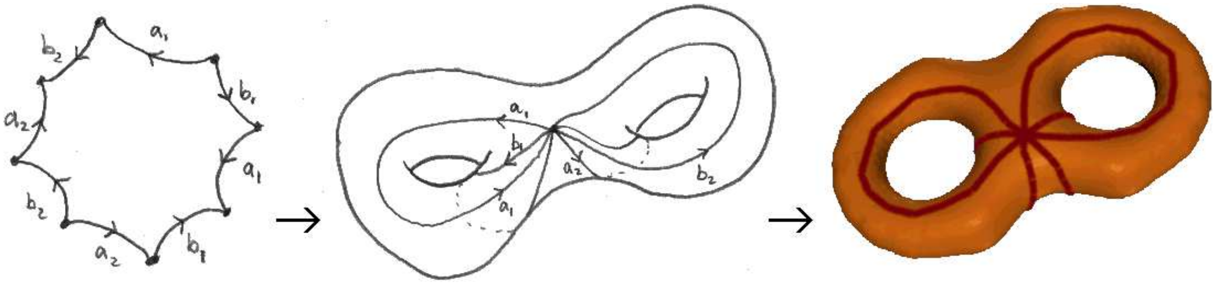

Figure 1: Surface of genus 2 constructed from something like $\vec{a}_1\vec{b}_1\bar{a}_1\bar{b}_1\vec{a}_2\vec{b}_2\bar{a}_2\bar{b}_2$.

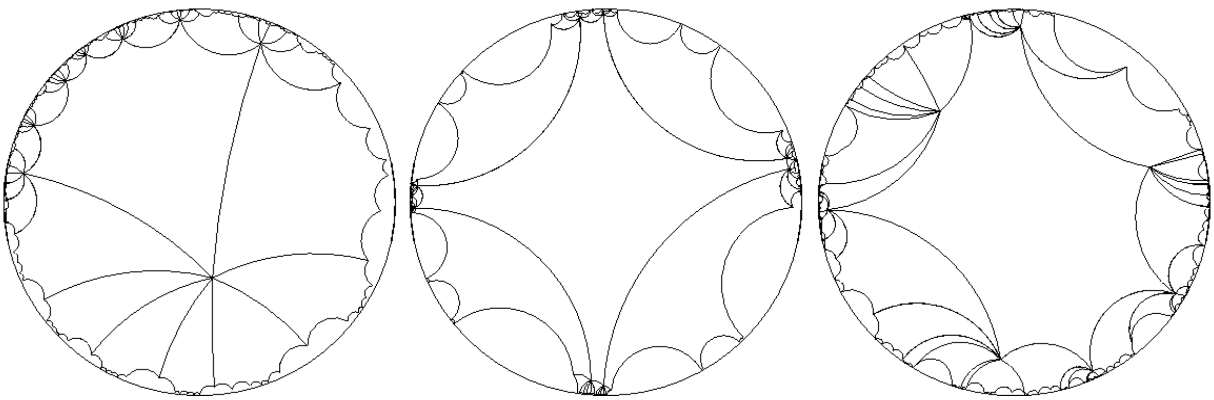

Figure 2: Three non-isometric ways to endow a surface of genus 2 with the hyperbolic metric. Pics courtesy of Deva Van Der Werf.

The Fenchel-Nielsen Coordinates of Teichmüller Spaces

“The Riemann Moduli Problem is to describe the isomorphism classes of all Riemann surfaces in a given topological class. Riemann solved the problem for simply connected Riemann surfaces: He found that the only possibilities are the disk, the plane, and the Riemann sphere. In the case of doubly connected Riemann surfaces, the problem is still not particularly difficult, although some technical complications arise, and they suggest that the problem should be clarified. The moduli problem becomes interesting for the more complicated Riemann surfaces. According to the Uniformization Theorem, these Riemann surfaces are covered by the upper half-plane H, and therefore are hyperbolic. While studying the moduli problem, Otto Teichmüller first proposed a modification of the moduli problem that gives rise to what we now call Teichmüller space. The **moduli space** of a Riemann surface R is the space of isomorphism classes of the complex structures on R; that is, the set of complex structures on R modulo the orientation preserving homeomorphisms of R. The **Teichmüller space** of R is a refinement of the moduli space, specifically the complex structures modulo homeomorphisms isotopic to the identity.” [K. Paur]

Riemann surfaces

A surface is by definition orientable. A **Riemann surface** is a surface R together with an **conformal structure**, i.e. a complete atlas where all transition maps $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cup U_\beta) \to \mathbb{C}$ are holomorphic. A **conformal metric** g on R in local coordinates is of the form $\lambda^2(z)\mathrm{d}z\mathrm{d}\bar{z}$.
Relevant examples of non compact surfaces carrying **hyperbolic metric** are $D := \{z \in \mathbb{C} : |z| < 1\}$ with $\lambda(z) = \frac{2}{1-|z|^2}$ and isomorphic $H := \{z \in \mathbb{C} : \mathrm{Im}\ z > 0\}$ with $\lambda(z) = \frac{1}{\mathrm{Im}\ z}$. The maps $z \in D \mapsto i\frac{1+z}{1-z} \in H$ and $z \in H \mapsto \frac{z-i}{z+i} \in D$ are isometries.
The group of **Möbius transformations** is isomorphic to the multiplicative group of matrices

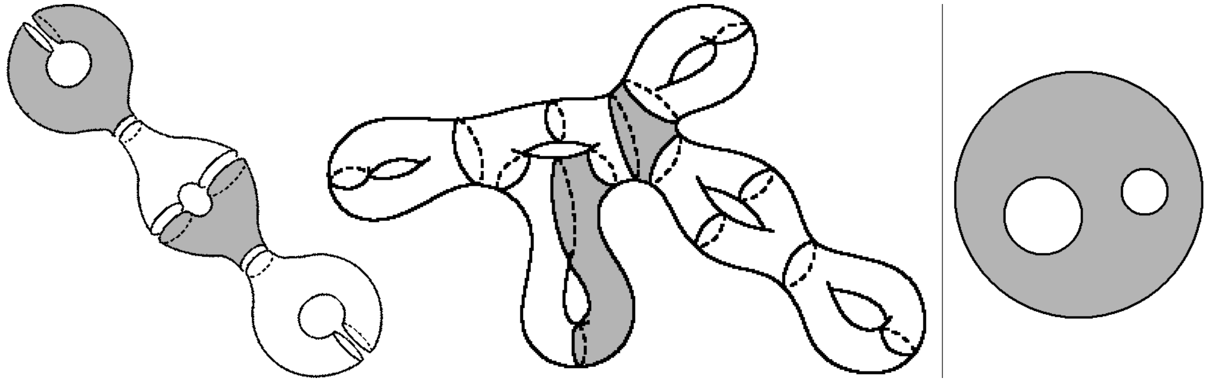

Figure 3: Decomposition of surfaces of genus $p \geq 2$ into Y-pieces. Pics courtesy of Jürgen Jost and Kathy Paur, resp. A default Y-piece as a subset of $\mathbb{C}$ is to the right.

$PSL(2,\mathbb{R}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a,b,c,d \in \mathbb{R}, ad-bc = 1 \right\} / \{\pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$, where each matrix defines an isometric transformation of H via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) \mapsto \frac{az+b}{cz+d}$.
A **Fuchsian model** of a Riemann surface R is the subgroup Γ of Möbius transformations obtained from the quotient $\Gamma = H/R$ under a covering map $H \to R$. Conversely, a properly discontinous subgroup Γ of $PSL(2,\mathbb{R})$ defines a Riemann surface via $R = H/\Gamma$. Some examples are shown in Figure 2.
A compact surface R with **genus** p is homeomorphic to p tori glued to a sphere. Hence the fundamental group is $\pi_1(R) = (\mathbb{Z}\times\mathbb{Z})^p$ with generators from $\vec{a}_1\vec{b}_1\bar{a}_1\bar{b}_1\vec{a}_2\vec{b}_2\cdots\vec{a}_p\vec{b}_p\bar{a}_p\bar{b}_p$.
R **carries** the hyperbolic metric if for all points $x \in R$ there is an isometry $\phi : U_x \to \phi(U_x) \subset H$ between open neighborhoods.
The **moduli space** $\mathcal{M}(R)$ of a Riemann surface is the set of hyperbolic metrics on the manifold R, where $(R, g_1) \sim (R, g_2)$ if there is an isometry between them.
The **Teichmüller space** $\mathcal{T}(R)$ is the set of triples (R, g, f), where g denotes the hyperbolic metric on R and $f : R \to R$ is a diffeomorphism under the following equivalence relation: $(R, g_1, f_1) \sim (R, g_2, f_2)$ if there exists an isometry k, so that $f_2 \circ f_1^{-1}$ and k are homotopic. Recall, that two maps $f_0, f_1 : R \to R$ are **homotopic**, if there is a continuous function $F : [0,1]\times R \to R$ with $F(0,\cdot) = f_0(\cdot)$ and $F(1,\cdot) = f_1(\cdot)$.

Surface R	$\pi_1(R)$	$\mathcal{M}(R)$	$\mathcal{T}(R)$
H	$\{0\}$	$\{0\}$	$\{0\}$
Y(see below)	$\mathbb{Z}^2$	$\mathbb{R}^3_+/\mathcal{S}_3$	$\mathbb{R}^3_+$
Torus	$\mathbb{Z}^2$	$H/PSL(2,\mathbb{Z})$	$\mathbb{R}_+ \times \mathbb{R}(= H)$
genus $p \geq 2$	$\mathbb{Z}^{2p}$	$\cdots$	$\mathbb{R}^{3p-3}_+ \times \mathbb{R}^{3p-3}$

Hyperbolic Y-pieces

Motivated by Figure 3 we aim to decompose a surface of genus $p \geq 2$ into subsets each isometric to a Y-piece with a suitable hyperbolic metric. For further considerations we choose the embedding
$Y := \{z \in \mathbb{C} : |z| \leq 1, \frac{1}{4} \leq |z - \frac{1}{2}|, \frac{1}{4} \leq |z + \frac{1}{2}|\}$. Let Y carry a hyperbolic metric, so that the three boundary curves c_1, c_2, c_3 are geodesics.

Lemma 1. For each $i \neq j$, there exists a unique shortest geodesic arc c_{ij} connecting c_i with c_j. The arc c_{ij} meets c_i with c_j orthogonally and has no self-intersections. Obviously $c_{ij} = c_{ji}$, but different c_{ij} do not intersect.

Incomplete proof: From Schüth's DiffGeo.II we have the intuition, that the distance function
$d : c_i \times c_j \to \mathbb{R}$ is locally convex and the shortest connecting path between two suitable points is a smooth geodesic path c_{ij}.
No self-intersections: Suppose that there exists $0 < t_1 < t_2 < 1$ with $c_{ij}(t_1) = c_{ij}(t_2)$. Then $c'_{ij} = c_{ij}|_{[0,t_1]} \cup c_{ij}|_{[t_2,1]}$ defines a curve connecting c_i and c_j, which is even shorter that c_{ij}. A contradiction.
Orthogonality: By the same argument as in Euclidean geometry, c_{ij} has to meet c_i and c_j orthogonally,

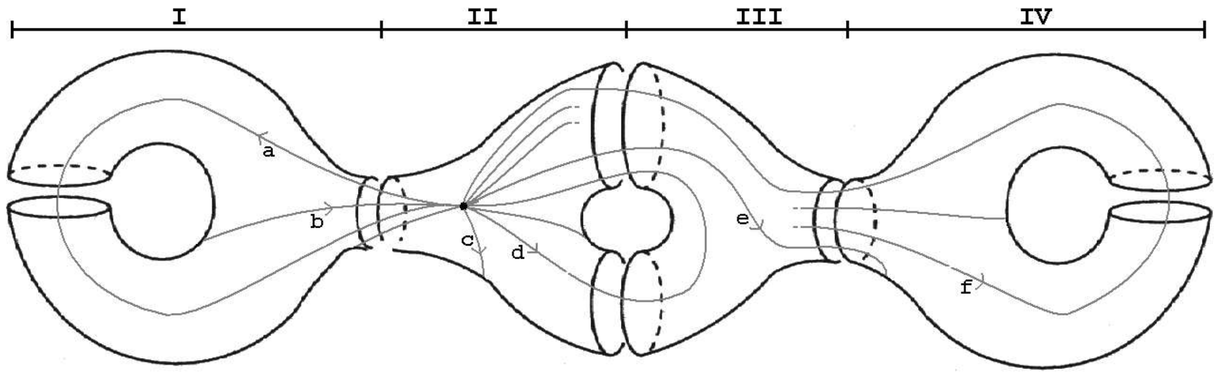

Figure 4: Surface of genus 3 composed by Y-pieces I,...,IV. $a, b, ..., f$ is a basis of the loop group π_1.

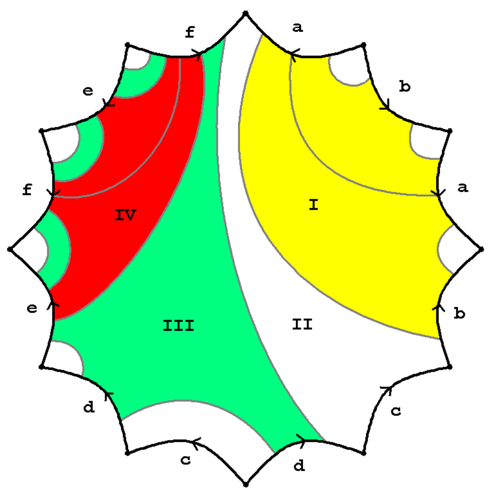

Figure 5: Surface of Figure 4 embedded in D as the disjoint union of Y-pieces I,...,IV.

as otherwise one could construct an even shorter curve joining c_i and c_j.
No pairwise intersections: Suppose there exist $t_1, t_2 \in (0,1)$ with $c_{ij}(t_1) = c_{ik}(t_2)$ for $j \neq k$. Assume wlog $\text{len}(c_{ij}|_{[0,t_1]}) \leq \text{len}(c_{ik}|_{[0,t_2]})$. Then $c'_{ik} = c_{ij}|_{[0,t_1]} \cup c_{ik}|_{[t_2,1]}$ connects c_i and c_k and satisfies $\text{len}(c'_{ik}) \leq \text{len}(c_{ik})$, hence is a shortest curve in its class, and is a smooth geodesic. Hence the initial curves c_{ij} and c_{ik} share tangent directions in the intersection point. A contradiction to the uniqueness of geodesics. Similarly, $c_{ij}(0) \neq c_{ik}(0)$ since they start at a right angle to c_i.
Uniqueness: Suppose there exists different curves c_{ij} and c'_{ij} connecting c_i and c_j. By the above considerations, we may assume they are disjoint. Therefore, there exists a geodesic quadrilateral in Y with all right angles. A contradiction to Gauss-Bonnet: A geodesic polygon P in H with k vertices and interior angles α_i has $\text{area}(P) = (k-2)\pi - \sum \alpha_i$. □

Lemma 2. For each $\lambda_1, \lambda_2, \lambda_3 > 0$, there exists a unique hyperbolic geodesic hexagon with sequent sides $a_1, b_1, a_2, b_2, a_3, b_3$ with $\text{len}(a_i) = \lambda_i$, and all right interior angles.

Proof. See Jürgen Jost Lemma 4.3.2 on pages 176-179. □

Theorem 3. The hyperbolic structure of a Y is uniquely determined by the lengths of c_1, c_2, c_3. For any $l_1, l_2, l_3 > 0$, there exists a Y with boundary curves possessing these lengths.

Proof. We split a given Y into two hexagons along the geodesic arcs c_{12}, c_{23}, c_{31} of Lemma 1. Both hexagons are isometric by Lemma 2. Hence, the three remaining sides have lengths $\frac{l_1}{2}, \frac{l_2}{2}, \frac{l_3}{2}$. Again by Lemma 2, these hexagons are uniquely determined from l_1, l_2, l_3 and so is their union Y.
Conversely, for fixed $l_1, l_2, l_3 > 0$, glue two unique isometric hexagons with lengths $\frac{l_1}{2}, \frac{l_2}{2}, \frac{l_3}{2}$ of alternating sides along the three remaining sides to form a Y. □

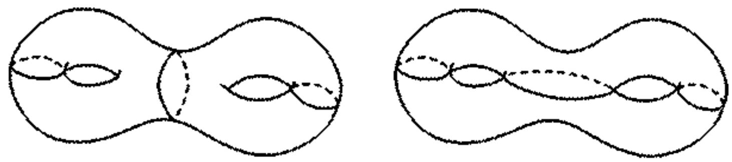

Figure 6: The decomposition is not unique?

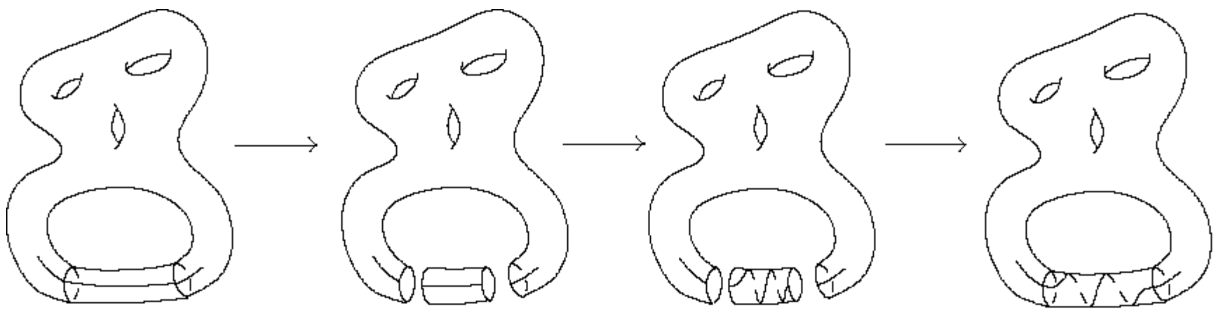

Figure 7: Visualization of Dehn twist by Kathy Paur.

Decomposition of surfaces of genus $p \geq 2$

We employ a decomposition of compact surfaces that does not generalize to the sphere or the torus, surfaces of genus < 2. Hence from hereon, let $R = H/\Gamma$ be a (compact) Riemann surface with genus $p \geq 2$.

Using combinatorics, cutting R along suitable $3p - p$ non-intersecting curves, the surface decomposes into $2p - p$ Y-pieces. The following theorem shows that we might as well cut along geodesics c_k with $k = 1, ..., 3p - 3$, but neither reference assures, that we may cut along pairwise disjoint geodesics c_k.

Theorem 4. Each closed curve $\gamma : S^1 \to R$ is homotopic to a unique closed geodesic $c : S^1 \to R$. If γ has no self-intersections, then c is likewise free from self-intersections.

Proof. Kathy Paur gives an elegant and rigorous proof. □

As seen in the previous section, the lengths $l_1, l_2, ..., l_{3p-3}$ of the c_k determine the hyperbolic metrics of the Y-pieces. When gluing the pieces back together, we notice that we may choose phase angles $\alpha_1, \alpha_2, ..., \alpha_{3p-3}$. As in the discussion of the hyperbolic torus, two phase displacements α_k, β_k result in isometric surfaces $R \iff \alpha_k - \beta_k \in 2\pi\mathbb{Z}$, note Figure 7. Furthermore, for two isometric surfaces there can only be an isometry homotopic to the identity, if $\alpha_k = \beta_k$.
Overall, for hyperbolic metrics on a surface of genus $p \geq 2$ modulo homeomorphisms isotopic to the identity, there are $2(3p-3)$ degrees of freedom in the coordinate space by **Fenchel-Nielsen** $\mathbb{R}_+^{3p-3} \times \mathbb{R}^{3p-3}$, which thus can be shown[1] is homeomorphic to $\mathcal{T}(R) = \mathcal{T}_p$.

References

Johann Sebastian Bach - The Art of Fugue
Jürgen Jost - Compact Riemann Surfaces, SK 750 J84
Kathy Paur - The Fenchel-Nielsen Coordinates of Teichmüller Space

[1] requires topology and metric on $\mathcal{T}_p$