Analysis of Quasi-Uniform Subdivision [1]

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ABSTRACT

We study the smoothness of quasi-uniform bivariate subdivision. A quasi-uniform bivariate scheme consists of different uniform rules on each side of the y-axis, far enough from the axis, some different rules near the y-axis, and is uniform in the y direction. For schemes that generate polynomials up to degree m, we derive a sufficient condition for C^m continuity of the limit function, which is simple enough to be used in practice. It amounts to showing that the joint spectral radius of a certain pair of matrices has to be less than 2^{-m}. We also relate the Hölder exponent of the m-th order derivatives to that joint spectral radius. The main tool is an extension of existing analysis techniques for uniform subdivision schemes, although a different proof is required for the quasi-uniform case. The same idea is also applicable to the analysis of quasi-uniform subdivision processes in higher dimension. Along with the analysis we present a 'tri-quad' scheme, which is combined of a scheme on a triangular grid on the half plane $x < 0$ and a scheme on a square grid on the other half plane $x > 0$ and special rules near the y-axis. Using the new analysis tools it is shown that the tri-quad scheme is globally C^2.

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1. INTRODUCTION

The smoothness analysis of subdivision schemes is mostly confined to the case of uniform schemes on uniform grids. In the uniform case there are several well established analysis tools such as the Fourier analysis approach (see e.g., [3, 4, 6, 7]), the z-transform tools (see e.g., [2, 9, 10]) using difference schemes and in terms of the joint spectral radius of the local subdivision operators (see e.g., [14, 15, 24]). A special non-uniform analysis is required in the analysis of subdivision schemes over meshes of general topology. In this case, there is a special structure and a special analysis around 'extraordinary vertices', using eigensystem analysis of the local subdivision operator, and using a special parametrization by the 'characteristic map' (see e.g., [8, 22, 23, 26]). Recently, non-uniform subdivision schemes have been analyzed, by extending the tools of difference schemes to non-uniform schemes over uniform grids [19] and to schemes over non-uniform grids [5, 13, 25]. In the present work we are interested in the analysis of quasi-uniform subdivision schemes. Such schemes may be of interest when matching two patches, where in each patch a different uniform subdivision scheme is applied, or in designing a scheme interpolating a curve on the surface (see [17, 16]). A univariate study of piecewise uniform schemes is presented in [11]. The analysis presented in this paper combines a few ideas of the above mentioned tools into a new method which is specially designed for quasi-uniform subdivision schemes. It combines eigensystem analysis with a joint spectral radius check and implicit divided differences considerations, and it also involves non-stationary matrix subdivision analysis. Along with the general discussion we consider a specific quasi-uniform scheme, the 'tri-quad' scheme, which is combined of Loop scheme on a triangular grid on the half plane $x < 0$ and of Catmull-Clark scheme on a square grid on the other half plane $x > 0$. A scheme of

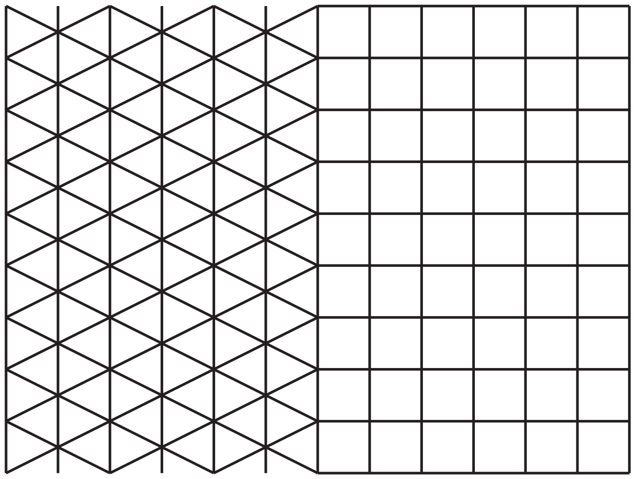

FIG. 1 The tri-quad grid.

this type has already been considered in [21], where the benefit in using 'tri-quad' meshes is explained. The particular scheme used in [21] is defined on meshes of general topology, composed of triangular and quadrilateral faces. It is not a C^2 scheme, yet it apparently produces limit surfaces with everywhere bounded curvatures. In this work, to properly define the special rules for the tri-quad scheme near the y-axis we employ a recent procedure suggested in [18]. The resulting tri-quad scheme accompanies the definitions and the assumptions of the general theory presented in the next section, and is used to demonstrate the analysis tools. It is shown that the new tri-quad scheme is globally C^2.

2. DEFINITIONS, ASSUMPTIONS AND THE TRI-QUAD SCHEME

We consider a *quasi-uniform grid* $X \in I\!R^2$, namely a grid which is uniform in each of the half planes, $x > 0$ and $x < 0$, and such that $\mathcal{E}X \equiv \{(i, j+1)|(i,j) \in X\} = X$, $2X \subset X$ and $\overline{\cup_{n=0}^{\infty} 2^{-n} X} = I\!R^2$. The leading example of a quasi-uniform grid in this paper is the tri-quad grid in Figure 1.

Let $l(X)$ denote the space of all control point sequences $l(X) = \{P \mid P : X \to \mathbb{R}\}$. The subdivision operator S is a linear operator on $l(X)$, $S : l(X) \to l(X)$. A stationary subdivision scheme is defined as the repeated application of S to given control points $P \in l(X)$.

We say that S is *convergent*, if for every $P \in l(X)$, there exists $F \in C(\mathbb{R}^2)$ (called the limit function) such that

$$\lim_{n\to\infty} \left\| S^n P - F\left(2^{-n}\cdot\right)\right\|_{\infty, X\cap 2^n\mathcal{D}} = 0, \tag{1}$$

for any open and bounded domain $\mathcal{D} \subset \mathbb{R}^2$. We denote $S^\infty P = F$. We also require, as part of the definition of uniform convergence, that $S^\infty P$ is non-zero for some P. Notice that although $S^n P$ is formally defined as a sequence over X, we associate the value $S^n P(x)$ for $x \in X$, with the value of the limit function at $2^{-n}x$, as implied by (1).

We say that S is C^m if $S^\infty P \in C^m(\mathbb{R}^2)$ for any $P \in l(X)$. Furthermore, we say that S is $C^{m+\alpha}$ if the m-th order derivatives of $S^\infty P$ are Hölder continuous of order α for any $P \in l(X)$.

A *quasi-uniform bivariate scheme* consists of different uniform rules on each side of the y-axis, far enough from the axis, some different rules near the y-axis, and is uniform in the y-direction. We assume, of course, that S is C^m continuous away from the y-axis, and that the bivariate scheme generates Π_m, the space of bivariate polynomials up to degree m. The last requirement implies the existence of an 'inverse' Q of S^∞ on Π_m. The important properties of Q are summarized in the following theorem, proved in [18]. To state the result we introduce the notion of *leading coefficient preservation*. We say that $Q : \Pi_m \to l(X)$ preserves leading coefficients if

$$f \in \Pi_k \Rightarrow |Qf(x) - f(x)| = o(\|x\|^k) \ , \quad as \quad \|x\| \to \infty \ , \ x \in X \ , \tag{2}$$

for all $k \leq m$. For example, any operator of the form $Qf(x) = f(x) + Df(x)$, where D is a linear differential operator, $D1 = 0$, preserves leading coefficients. Here, for $P \in l(X)$ and $x \in X$, $P(x)$ denotes the the entry of P attached to x. We also introduce the *dilation operator* σ,

$$\sigma f = f\left(\frac{\cdot}{2}\right) \ .$$

THEOREM 2.1. *[18] If S is a convergent subdivision scheme, S^∞ is an injection, and $Q : \Pi_m \to l(X)$ preserves leading coefficients, then*

$$SQf = Q\sigma f \ , \quad \forall f \in \Pi_m \ , \tag{3}$$

if and only if

$$S^\infty Qf = f \ , \quad \forall f \in \Pi_m \ . \tag{4}$$

Theorem 2.1 reduces (4), which is the formal notation for polynomial generation, to the condition (3), in which S appears as a linear term. This is useful for the construction of new subdivision schemes. Once we fix Q, condition (3) can be translated into a system of linear equations, from which we deduce the subdivision weights. This technique is demonstrated in [18], and is used in the following construction of the tri-quad scheme.

From (3) we also get important information about the eigenvalues and the eigenvectors of S. Considering a monomial $f = x^i y^j$, with $i + j \leq m$, it follows that $\sigma f = 2^{-(i+j)} f$ and thus

$$SQ\{x^i y^j\} = Q\sigma\{x^i y^j\} = 2^{-(i+j)} Q\{x^i y^j\} \ , \quad i + j \leq m \ . \tag{5}$$

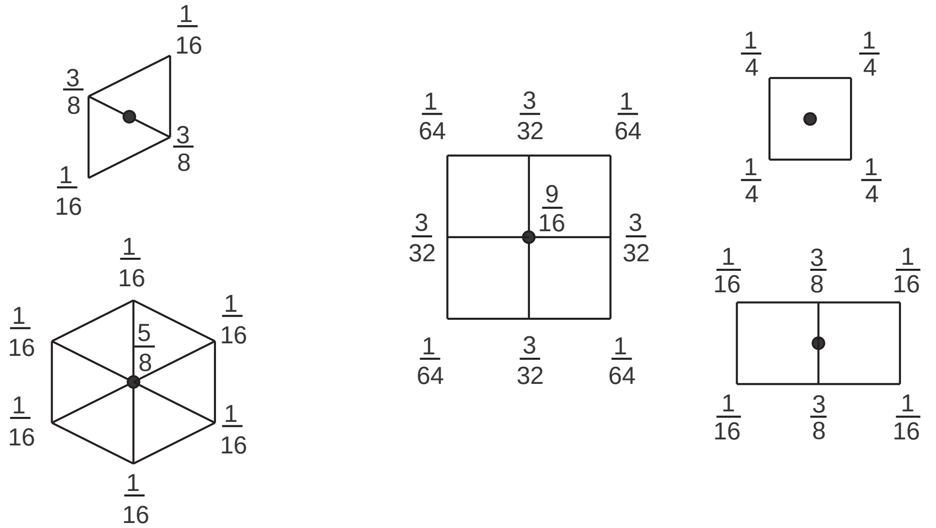

FIG. 2 The scheme masks away from the y-axis: Catmull-Clark scheme on the right and Loop scheme on the left.

I.e., $Q\{x^i y^j\}$ is an eigenvector of the scheme for $i+j \leq m$ with eigenvalue $2^{-(i+j)}$. Some examples of the operator Q for different subdivision schemes are given in [18].

Example 2.1 (The tri-quad scheme - construction). Considering the *tri-quad grid* in Figure 1, we would like to define a quasi-uniform scheme over this grid which is the the tensor product cubic B-spline scheme, or the Catmull-Clark scheme [1], on the right half plane, and the C^2 quartic three-directional box-spline scheme, or the Loop scheme [20], on the left half plane. The masks of these schemes are depicted in Figure 2.

The goal is to define special rules on the y-axis and near it so that overall the scheme will be C^2, i.e., as smooth as the right and left schemes. These special rules are constructed together with an operator Q, which also requires a special definition near the y-axis, so that the condition $SQ = Q\sigma$ holds for Π_2 over the entire plane. The operator Q away from the y-axis is defined as the appropriate Q

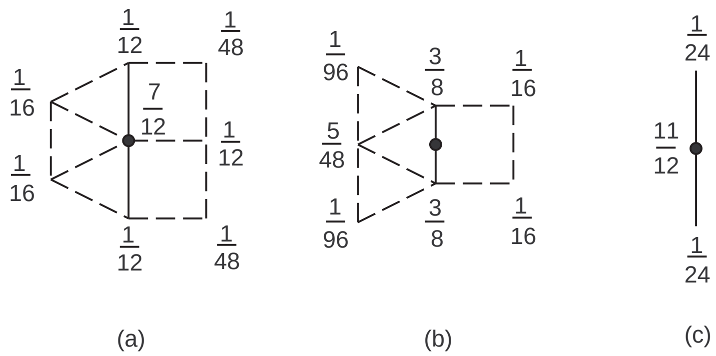

FIG. 3 The scheme near the y-axis: (a) The stencil for a new value at old grid points on the y-axis. (b) The stencil for a new value at new grid points on the y-axis. (c) The stencil of the operator defining temporary values on the y-axis before the application of the Loop scheme on $x < 0$.

operator for the right and left uniform schemes, i.e.,

$$Qf = Q^+ f = f - \frac{1}{6} f_{xx} - \frac{1}{6} f_{yy} \ , \quad x \geq 0 \ ,$$

$$Qf = Q^- f = f - \frac{1}{6} f_{xx} - \frac{1}{8} f_{yy} \ , \quad x < 0 \ .$$

It is easy to verify that Q^+ and Q^- satisfy the required equation (3), with $m = 2$, for the right and left schemes respectively. Given this choice of Q, the special subdivision rules near the y-axis are defined by requiring the conditions (5), for $m = 2$. The equations coming out of (5) are solvable, but not uniquely. The challenge is to find a scheme of the smallest possible support which fulfills (5). A scheme with positive weights and of small support, though probably not the smallest possible, is described by the rules shown in Figure 3. Note that the convolution stencil (c) is only used for calculating temporary values before the application of the uniform left scheme.

By Theorem 2.1 it then follows that this scheme generates polynomials in Π_2. It is now left to be shown that this scheme generates C^2 limit functions over the entire plane. We note that it is possible to define a scheme that generates polynomials up to degree 3, but this cannot improve the smoothness beyond C^2.

Remark 2.2. The choice $Q = Q^+$ on the y-axis is somewhat arbitrary. Different choices of Q lead to different subdivision rules. By experimenting with other choices of Q on the y-axis, we found that for some of them there does not exist subdivision schemes S with positive weights. (e.g. $Q = Q^-$ or $Q = \frac{Q^- + Q^+}{2}$ on the y-axis). With $Q = Q^+$ on the y-axis we were able to get a subdivision scheme that consists of only three special rules, in which all weights are positive.

3. THE ANALYSIS PROCEDURE AND THE TRI-QUAD EXAMPLE

In the following, we describe the procedure for checking whether a given quasi-uniform scheme S is C^m. We assume that S generates polynomials up to degree m, in the sense that (3) is satisfied for some Q. The justification of the different steps is given in the following sections.

First we recall (see [25]) that the local subdivision matrix that maps a region around the origin to itself must satisfy the necessary conditions for C^m smoothness. Namely, that its eigenvalues are $(1, 0.5, 0.5, \ldots, 2^{-m}, \ldots, 2^{-m})$ and each of them corresponds to an eigenvector with a polynomial as the limit function. The rest of the eigenvalues must be strictly smaller than 2^{-m}.

The analysis procedure:

1. Let $L \subset X$ denote a subset of mesh points around the origin such that the values of the limit function in $[-1, 1] \times [0, 1]$ depend only on control points in L. Furthermore, the values at iteration 1 in L and in $\mathcal{E}L$, namely $SP|_L$

and $SP|_{\mathcal{E}L}$, depend only on the initial values in L, $P|_L$, where $\mathcal{E}$ is a shift operator, $\mathcal{E}L = \{(i, j+1)|(i,j) \in L\}$.

2. Let A denote the local subdivision operator taking values in L to values in L after one subdivision iteration. Let B denote the operator taking values in L to values in $\mathcal{E}L$.

3. Using the left and right eigenvectors of A, form a basis V for the vectors of values in L such that the matrix form of A in the new basis is

$$\widetilde{A} = \left[\begin{array}{c|c} \Lambda & C_0 \\ \hline 0 & Y_0 \end{array} \right], \tag{6}$$

Where $\Lambda = diag(1, 0.5, 0.5, ..., 2^{-m}, ..., 2^{-m})$. One way to do it is to compose the basis V from the $(m+1)(m+2)/2$ right eigenvectors,

$$Qf|_L \ , \quad f = x^i y^j \ , \quad 0 \le i+j \le m \ , \tag{7}$$

and a basis of the null space of the corresponding left eigenvectors.

4. From the polynomial generation assumption about the scheme, it turns out that the matrix form of B in the basis V is

$$\widetilde{B} = \left[\begin{array}{c|c} \Theta & C_1 \\ \hline 0 & Y_1 \end{array} \right], \tag{8}$$

where Θ is an upper-triangular matrix that has the same diagonal as Λ. Moreover, Θ has certain zero values above the diagonal, creating such diagonal blocks of sizes $1, 2, 3, 4, \ldots$, e.g., for $m = 2$

$$\Theta = \left[\begin{array}{c|cc|ccc} 1 & * & * & * & * & * \\ \hline 0 & 0.5 & 0 & * & * & * \\ 0 & 0 & 0.5 & * & * & * \\ \hline 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 \end{array}\right]. \tag{9}$$

5. A sufficient condition for C^m continuity is that the *joint spectral radius* of Y_0 and Y_1, $\rho_\infty(Y_0, Y_1)$, is strictly less than 2^{-m}, where

$$\rho_\infty(Y_0, Y_1) =$$
$$\lim_{k \in \mathbb{Z}_+ \setminus 0} \left(\max \left\{ \|Y_{\varepsilon_k} Y_{\varepsilon_{k-1}} \cdots Y_{\varepsilon_1}\|_\infty : \varepsilon_i \in \{0, 1\}, i = 1, \ldots, k \right\} \right)^{\frac{1}{k}} . \tag{10}$$

   Moreover, if $\rho_\infty(Y_0, Y_1) = 2^{-(m+\alpha)}$, $0 < \alpha \leq 1$ then the m-th order derivatives of the limit function are Hölder continuous with exponent $\alpha - \epsilon$ for arbitrarily small $\epsilon > 0$. Of course, this only holds if the limit function away from the y-axis is known to have that Hölder exponent.

Remark 3.1. Notice that the limit of the sequence in (10) always exists. Let

$$\rho_\infty^{[k]}(Y_0, Y_1) = \left(\max \left\{ \|Y_{\varepsilon_k} Y_{\varepsilon_{k-1}} \cdots Y_{\varepsilon_1}\|_\infty : \varepsilon_i \in \{0, 1\}, i = 1, \ldots, k \right\} \right)^{\frac{1}{k}} . \tag{11}$$

Fix $k > 0$. For every $n > k$, let $n = k \cdot m + r$, where $0 \leq r < k$. It is easy to see that

$$\rho_\infty^{[n]}(Y_0, Y_1) \leq \rho_\infty^{[r]}(Y_0, Y_1)^{\frac{r}{n}} \rho_\infty^{[k]}(Y_0, Y_1)^{\frac{k \cdot m}{n}},$$

therefore,

$$\limsup_{n \in \mathbb{Z}_+ \setminus 0} \rho_\infty^{[n]}(Y_0, Y_1) \leq \rho_\infty^{[k]}(Y_0, Y_1), \quad \forall k \geq 0.$$

Since all sequence elements are greater or equal to the limsup, the limit exists. In particular, we can compute an upper bound for the joint spectral radius $\rho_\infty(Y_0, Y_1)$, by estimating the norms of all possible products of finite length k of Y_0 and Y_1. I.e.,

$$\rho_\infty(Y_0, Y_1) \le \rho_\infty^{[k]}(Y_0, Y_1). \tag{12}$$

Remark 3.2. The condition $\rho_\infty(Y_0, Y_1) < 2^{-(m+\alpha)}$, in view of the special basis V used in (6), implies that the mth degree Taylor expansion coefficients of $S^\infty P$ at dyadic points on the y-axis are all uniformly bounded. This is the main idea behind the theory presented here, the detailed proof is presented in §6.

EXAMPLE 3.3 (The tri-quad scheme - C^2 analysis).

Let us apply the above analysis tools for the tri-quad scheme presented above. The set L is the set of $|L| = 45$ points

$$L = \{(i,j) : i = 0,1,2\ ,\ -4 \le j \le 4\ ,\ j \in \mathbb{Z}\} \cup$$
$$\{(i, j + 0.5i) : i = -1,-2\ ,\ -4 \le j \le 4\ ,\ j \in \mathbb{Z}\}\ .$$

The matrices A and B are evaluated as follows: First we choose an ordering of the points in L, $L = \{(i_1, j_1), \cdots, (i_{|L|}, j_{|L|}\}$. An entry $A_{k,\ell}$ in A corresponds to a pair of points $((i_k, j_k), (i_\ell, j_\ell))$. Applying the subdivision scheme to initial data set $P = \delta_{(i_\ell, j_\ell)}$ which is 1 at the point (i_ℓ, j_ℓ) and zero elsewhere, we have

$$A_{k,\ell} = (S\delta_{(i_\ell, j_\ell)})_{(i_k, j_k)},\quad k = 1, \cdots, |L|,\quad \ell = 1, \cdots, |L|.$$

The entries of the matrix B are

$$B_{k,\ell} = (S\delta_{(i_\ell, j_\ell)})_{(i_k, j_k+1)},\quad k = 1, \cdots, |L|,\quad \ell = 1, \cdots, |L|.$$

The matrices $\widetilde{A}$ and $\widetilde{B}$ are just the representation of A and B respectively in another basis V. The construction of this basis is described in item 3 of the analysis procedure above, and it involves the computation of the polynomial eigenvectors of S by (7).

The upper-left block Θ of $\widetilde{B}$ for the tri-quad scheme is

$$\Theta = \left[\begin{array}{c|cc|ccc} 1 & -0.1859 & 0.0476 & -0.0039 & 0.0271 & -0.0181 \\ \hline 0 & 0.5 & 0 & -0.0036 & -0.1398 & 0.0921 \\ 0 & 0 & 0.5 & -0.0968 & 0.0241 & -0.0216 \\ \hline 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 \end{array}\right]. \tag{13}$$

A bound for $\rho_\infty(Y_0, Y_1)$ may be estimated by $\rho_\infty^{[k]}(Y_0, Y_1)$ using Remark 3.1, and this is used to compute a lower estimate $\alpha_k = -2 - log_2(\rho_\infty^{[k]}(Y_0, Y_1))$ of the Hölder exponent. We obtained

$$\alpha \geq \alpha_{18} = -2 - log_2(\rho_\infty^{[18]}(Y_0, Y_1)) = 0.5942. \tag{14}$$

Hence, we deduce that the tri-quad scheme is at least $C^{2.5942}$. A straightforward extrapolation of the values α_k as a function of $1/k$ indicates that $lim_{k\to\infty}\alpha_k \sim 1$, leading to the conjecture that the tri-quad scheme is $C^{3-\epsilon}$ for any $\epsilon > 0$. This conjecture is, at least, in agreement with the spectral radii of Y_0 and Y_1, $\rho(Y_0) = \rho(Y_1) = \frac{1}{8}$.

The following sections justify the above analysis procedure.

4. THE MATRIX SUBDIVISION SCHEME (A, B)

By assumption, the subdivision scheme is C^m away from the y-axis, and all we need is to check the convergence and the smoothness near the y-axis. We do it by monitoring the values generated in a wide enough strip of mesh points along the

y-axis. Specifically, we consider the strip

$$J = \{(i,j)\ :\ -M \le i \le N\ ,\ \ j \in \mathbb{Z}\ \}\ =\ \bigcup_{j\in\mathbb{Z}} \mathcal{E}^j L\ . \tag{15}$$

By the definition of L, the subdivision scheme S takes values in J to values in J in the next iteration,

$$Sf^n|_J = f^{n+1}|_J\ .$$

Restricted to J, we view S as a univariate vector-valued subdivision scheme, as follows:

To each integer point $(0, j2^{-n})$ on the y-axis we relate the vector of data values $v_j^n = f^n|_{\mathcal{E}^j L}$. The operators A and B defined above constitute a vector-valued binary subdivision scheme with the 2-term mask (A,B), namely,

$$v_{2j}^{n+1} = Av_j^n\ ,\qquad v_{2j+1}^{n+1} = Bv_j^n\ . \tag{16}$$

This scheme is equivalent to S near the y-axis, in the following sense. First, every set of control points generated by S can be generated by the vector-valued scheme, simply by taking as initial data to the vector-valued scheme, groups of control points over integer shifts of L in the y-direction, $v_j^0 = P|_{\mathcal{E}^j L}$. Second, the values $\{v_j^n\}_{j=0}^{2^n-1}$ at iteration n of the vector scheme are the same as the values generated by S at the n-th iteration if the initial data for S over L is taken as v_0^0.

Also, by the definition of L, the strip J is wide enough to capture the behavior of the mth order derivatives on the y-axis. If we show that the vector-valued scheme with the mask (A,B) generates bounded sequences, then we know that S generates bounded values. If the vector-valued scheme is C^0, i.e., generates C^0 univariate vector-valued functions, then it follows that S is C^0 along the y-axis as well. Yet, the mask of the vector-valued scheme has only two terms, and, as such, it cannot produce a C^0 limit from an arbitrary vector-valued data. Even if we find a way to overcome this hurdle, we cannot push it further to higher order smoothness, since the vector-valued scheme is univariate and we are also interested in derivatives in the x-direction and in mixed derivatives as well.

Here comes into play the representation of the vector-valued data in the eigenvectors basis. The idea here is, that knowing the coefficients in the eigenvectors expansion at a point, gives us the Taylor expansion (up to degree m) of the limit function at that point. I.e., the coefficient of the monomial eigenvector $Q[x^i y^j]$ with eigenvalue $2^{-(i+j)} \geq 2^{-m}$ is the coefficient of $x^i y^j$ in the Taylor expansion. The eigenvectors corresponding to eigenvalues smaller than 2^{-m} do not contribute to the mth order derivatives. At a given dyadic point we know exactly how the coefficients of the monomial eigenvectors evolve with the iterations. The matrix subdivision scheme with the mask $(\widetilde{A}, \widetilde{B})$ fills up the coefficients of these eigenvectors on finer grids.

We want to show that the coefficients corresponding to the main eigenvalues remain bounded or tend to zero at a certain rate during the refinement process. For example, the constant term must remain bounded. Linear terms at refinement level n, multiplied by 2^n must remain bounded. In general, the coefficient of the monomial eigenvector $Q[x^i y^j]$ corresponding to the monomial $x^i y^j$, should stay bounded when multiplied by $2^{(i+j)n}$. All the rest of the coefficients of eigenvectors must tend to zero when multiplied by $2^{n(m+\alpha)}$, $\alpha > 0$. If these conditions are satisfied, we can show that the m-th order derivatives of S are Hölder continuous of order α.

In order to study the rate at which certain coefficients tend to zero, we rescale the vector-valued scheme $(\widetilde{A}, \widetilde{B})$, multiplying by corresponding powers of 2^n when represented in the basis V. We obtain a non-stationary vector-valued scheme

(A_n, B_n) where the masks (A_n, B_n) depend on the iteration level n, through

$$A_n = \Delta^{n+1}\widetilde{A}\Delta^{-n}, \quad B_n = \Delta^{n+1}\widetilde{B}\Delta^{-n},$$

where Δ is a diagonal matrix with the diagonal values

$$diag(\Delta) = (1, 2, 2, 4, 4, 4, \ldots, 2^m, \ldots, 2^m, 2^{m+\alpha}, \ldots, 2^{m+\alpha}). \tag{17}$$

Our goal, then, is to show that the non-stationary vector-valued scheme is *stable*, i.e. it generates values which are uniformly bounded, given bounded initial data.

It is easy to see that A_n converges geometrically to a matrix with the shape

$$A_\infty = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 2^{m+\alpha}Y_0 \end{array}\right].$$

Also, due to the shape of Θ (9), B_n also converges with the same speed to

$$B_\infty = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 2^{m+\alpha}Y_1 \end{array}\right].$$

The particular issue of the stability of non-stationary subdivision schemes has been studied in [12]. It turns out that the non-stationary scheme (A_n, B_n) is asymptotically equivalent to the limit scheme (A_∞, B_∞), and thus it is enough to check whether (A_∞, B_∞) is stable. A necessary condition for the stability of the scheme (A_∞, B_∞) is that the joint spectral radius of Y_0 and Y_1, $\rho_\infty(Y_0, Y_1)$, does not exceed $2^{-(m+\alpha)}$.

5. C^m ANALYSIS AND HÖLDER CONTINUITY NEAR THE y-AXIS

In this section we relate the uniform rate of decay of coefficients of eigenvectors of A to the Hölder exponent of continuity of the m-th order derivatives of the limit function. We suppose that the limit function away from the y-axis is in $C^{m+\alpha}$, i.e., its m-th order derivatives away from the y-axis have Hölder exponent of continuity $0 < \alpha \leq 1$.

Let us denote the Hölder constant of a function F in a domain $U \subset \mathbb{R}^2$ by

$$H(F, \alpha, U) = \sup_{x,y \in U, x \neq y} \frac{|F(x) - F(y)|}{\|x - y\|^\alpha}. \tag{18}$$

We define a domain W as the pair of rectangles

$$W = ([-1, -\frac{1}{2}] \times [0, 1]) \cup ([\frac{1}{2}, 1] \times [0, 1]) \ . \tag{19}$$

By the definition of L, the limit function on W depends only on the control points in L. Assuming that the m-th order derivatives away from the y-axis are Hölder continuous with Hölder exponent α, we get, from the linearity and the local support of S, that

$$H(D^m S^\infty P, \alpha, W) \leq c\|P\|_{\infty,L}, \tag{20}$$

for some $c > 0$, where D^m denotes any differential operator D^m of order m. It is also easy to verify that for any domain U,

$$H(F(\lambda \cdot), \alpha, U) = \lambda^\alpha H(F, \alpha, \lambda U), \quad \forall \lambda > 0. \tag{21}$$

We want to study the Hölder constant of $S^\infty P$ closer and closer to the y-axis, $H(D^m S^\infty P, \alpha, 2^{-n} W)$, $n \in \mathbb{Z}_+$. But $S^\infty P = (S^\infty S^n P)(2^n \cdot)$. Therefore, we get using (21) that

$$\begin{aligned} H(D^m S^\infty P, \alpha, 2^{-n} W) &= H(2^{mn} D^m (S^\infty S^n P)(2^n \cdot), \alpha, 2^{-n} W) \\ &= 2^{mn} 2^{\alpha n} H(D^m S^\infty S^n P, \alpha, W), \end{aligned}$$

and then from (20), we have

$$H(D^m S^\infty P, \alpha, 2^{-n} W) \leq 2^{n(m+\alpha)} c\|S^n P\|_{\infty,L}. \tag{22}$$

Similarly, the limit function on $2^{-n}\mathcal{E}^i W$ depends only on the values of $S^n P$ in $\mathcal{E}^i L$, and by using (21) and recalling that S is y-direction shift invariant, we get

$$H(D^m S^\infty P, \alpha, 2^{-n}\mathcal{E}^i W) \leq 2^{n(m+\alpha)} c \|S^n P\|_{\infty,\mathcal{E}^i L} \ , \quad \forall n \in \mathbb{Z}_+ \ , \quad \forall i \in \mathbb{Z} \ . \tag{23}$$

Later on we use the above relations to prove that the Hölder constant of $D^m S^\infty P$ is uniformly bounded in the domains $2^{-n}\mathcal{E}^i W$, $n \in \mathbb{Z}_+$, $i \in \mathbb{Z}$. In the next section we show that uniform Hölder continuity over the domains $2^{-n}\mathcal{E}^i W$ implies Hölder continuity over the entire plane.

6. UNIFORM HÖLDER CONTINUITY OVER THE PLANE

We now show how to deduce Hölder continuity over a domain from the uniform Hölder continuity over subsets of the domain, provided that the closure of their union covers the domain.

Since we assume Hölder continuity away from the y-axis, we restrict our attention to the strip $[-1, 1] \times [-\infty, \infty]$. Define

$$U = \bigcup_{j \in \mathbb{Z}} \mathcal{E}^j W = ([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]) \times (-\infty, \infty) \ .$$

LEMMA 6.1 (Uniform Hölder continuity away from the y-axis). *Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ denote a function which is continuous everywhere except maybe the y-axis. If for all $i \in \mathbb{Z}$, $H(F, \alpha, \mathcal{E}^i W) \leq c$, then for any $p = (p_1, p_2), q = (q_1, q_2) \in U$ such that $|p_2 - q_2| \leq 1$,*

$$|F(p) - F(q)| \leq 3c\|p - q\|^\alpha \ . \tag{24}$$

Proof. Let $p, q \in U$, and let r denote the point with coordinates $(p_1, q_2) \in U$. Since r and q only differ by their first coordinate, they belong to the same integer shift of W, and therefore

$$|F(r) - F(q)| \leq c\|r - q\|^\alpha. \tag{25}$$

We now observe $|F(p)-F(r)|$. In case p and r belong to the same integer shift of W, we have that $|F(p)-F(r)| \le c\|p-q\|^{\alpha}$. Otherwise, we assume, w.l.o.g, that $r_2 > p_2$. We now use the assumption that $|p_2-q_2| \le 1$. For $s=(p_1, \lfloor r_2 \rfloor)$, we have that

$$\begin{aligned} |F(p)-F(s)| &\le c\|p-s\|^{\alpha}, \\ |F(r)-F(s)| &\le c\|r-s\|^{\alpha}. \end{aligned}$$

And, because $\|p-s\|, \|r-s\| \le \|p-r\|$, we get

$$|F(p)-F(r)| \le 2c\|p-r\|^{\alpha}. \tag{26}$$

From (25) and (26) we get that

$$|F(p)-F(q)| \le 3c\|p-q\|^{\alpha}.$$

∎

Corollary 6.2. *Let $n>0$. If for all $i \in \mathbb{Z}$, $H(F,\alpha,2^{-n}\mathcal{E}^i W) \le c$, then*

$$|p_2-q_2| \le 2^{-n} \Rightarrow |F(p)-F(q)| \le 3c\|p-q\|^{\alpha}, \quad \forall p,q \in 2^{-n}U.$$

Proof. Using (21), we have

$$H(F(2^{-n}\cdot),\alpha,\mathcal{E}^i W) \le 2^{-n\alpha}c.$$

From Lemma 6.1 it follows that $\forall p,q \in U$ such that $|2^n p_2 - 2^n q_2| \le 1$,

$$|F(2^{-n}2^n p) - F(2^{-n}2^n q)| \le 3 \cdot 2^{-n\alpha} c |2^n p - 2^n q|^{\alpha} = 3c\|p-q\|^{\alpha}.$$

∎

Lemma 6.3 (Uniform continuity near the y-axis). *Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ denote a function which is bounded and continuous everywhere except maybe on the y-axis.*

If for all $i \in \mathbb{Z}$, and $n \geq 0$, $H(F, \alpha, 2^{-n}\mathcal{E}^i W) \leq c$, then

$$H\left(F, \alpha, ([-1,1] \setminus \{0\}) \times (-\infty, \infty)\right) < \infty \ . \tag{27}$$

Furthermore, F may be redefined on the y-axis so that

$$H\left(F, \alpha, [-1,1] \times (-\infty, \infty)\right) < \infty \ . \tag{28}$$

Proof. For $p, q \in ([-1,1] \setminus \{0\}) \times (-\infty, \infty)$ we would like to show that $|F(p) - F(q)| \leq \tilde{c}|p - q|^\alpha$. This is established by considering all the different cases of relative locations of $p = (p_1, p_2)$ and $q = (q_1, q_2)$:

1. If p and q are far away from each other, we use the fact that F is bounded.

2. If p and q are not on the same side of the y-axis, we define $r = (-p_1, p_2)$, and use the triangle inequality $|F(p) - F(q)| \leq |F(p) - F(r)| + |F(r) - F(q)|$. $|F(p) - F(r)|$ can be bounded by Corollary 6.2 since p and r lie in the same strip $2^{-n}U$. The term $|F(r) - F(q)|$ will be estimated by cases 3,4,5 below.

3. The case $p_1 \neq q_1$ and $p_2 \neq q_2$, when p and q are from the same side of the y-axis. We define $r = (p_1, q_2)$ and use $|F(p) - F(q)| \leq |F(p) - F(r)| + |F(r) - F(q)|$. That reduces the problem to the cases $p_1 = q_1$ or $p_2 = q_2$.

4. The case $p_1 = q_1$. If $|p_2 - q_2| \leq \frac{1}{2}|p_1|$, Corollary 6.2 does the job. Otherwise, we define r_1 further from the y-axis, and use $|F(p) - F(q)| \leq |F(p_1, p_2) - F(r_1, p_2)| + |F(r_1, p_2) - F(r_1, q_2)| + |F(r_1, q_2) - F(q_1, q_2)|$. The mid-term is bounded using Corollary 6.2. The other terms are settled by case 5.

5. The case of p and q being on the same side of the y-axis and $p_2 = q_2$. This case is established by defining intermediate points along the line segment between p and q on the boundaries between dilations of U, and summing

up the contributions. If $p \in 2^{-m}U$ and $q \in 2^{-n}U$, $n > m$, we have, using Corollary 6.2,

$$|F(p) - F(q)| \leq 6c\|p - q\|^\alpha + 3c \sum_{m+1}^{n-1} (2^{-i} - 2^{-i-1})^\alpha \ ,$$

where the above sum is set to zero if $n = m + 1$. If $n > m + 1$ we have

$$\sum_{m+1}^{n-1} (2^{-i} - 2^{-i-1})^\alpha = \frac{1}{2^\alpha - 1}(2^{-(m+1)\alpha} - 2^{-n\alpha}) \leq \hat{c}\|p - q\|^\alpha \ .$$

The above considerations prove (27), i.e. uniform Hölder continuity over the split domain $\widetilde{U} = ([-1, 1] \setminus \{0\}) \times (-\infty, \infty)$. To extend the result to $[-1, 1] \times (-\infty, \infty)$ we observe that (27) implies that for any sequence $\{p^{(n)}\} \subset \widetilde{U}$ converging to a point $(0, y)$ on the y-axis, there is a unique limit $\lim_{n\to\infty} F(p^{(n)})$. The result (28) thus follows by redefining $F(0, y)$ as this limit. ∎

All the above results lead to the main result of this paper:

Theorem 6.4. *Assume that S is a C^m scheme away from the y-axis, and that the m-th order derivatives of its limit function there have Hölder exponent of continuity $0 < \alpha \leq 1$. Also, assume that (3) is satisfied for some $Q : \Pi_m \to l(X)$. Let Y_0 and Y_1 be defined as in* (6),(8).

If $\rho_\infty(Y_0, Y_1) < 2^{-(m+\alpha)}$, then S is globally C^m and the m-th order derivatives of its limit functions have Hölder exponent α.

Proof. First we note, that in order to avoid the problem of unbounded sequences of control points, it is enough to assume that the control points P are zero in $J \setminus L$. In view of Lemma 6.3, and since $D^m S^\infty P$ exists away from the y-axis, it suffices to show that $H(D^m S^\infty P, \alpha, 2^{-n}\mathcal{E}^i W) \leq c$, $\forall n \in \mathbb{Z}_+$, $i \in \mathbb{Z}$. Equation (23) exhibits the relation between the Hölder constant over the domains $2^{-n}\mathcal{E}^i U$ and the values of the control points over $\mathcal{E}^i L$, namely $\|S^n P\|_{\infty,\mathcal{E}^i L}$. It seems that we have to show that $\|S^n P\|_{\infty,\mathcal{E}^i L} = O\left(2^{-n(m+\alpha)}\right)$, which in general is false, so we have to be more careful.

Let G denote the projection of values in L onto the subspace of the $(m+1)(m+2)/2$ right eigenvectors of A, namely, $span\{\ Qf|_L\ ,\ \ f = x^i y^j\ ,\ \ 0 \le i+j \le m\}$. All the other eigenvectors of A are in $\ker(G)$. We note that $G(S^n P|_{\mathcal{E}^i L})$ consists only of a combination of eigenvectors that correspond to polynomials of degree $\le m$ in the limit. Their m-th order derivatives are either zero or constant, and they have zero Hölder constant. Therefore, we can reduce the discussion to data $S^n P|_{\mathcal{E}^i L}$ which is a combination of eigenvectors of A with eigenvalues smaller than 2^{-m}, i.e., to $(I-G)(S^n P|_{\mathcal{E}^i L})$. It is easy to check, in view of the definition of the matrix subdivision scheme (A, B), and in view of §4, that

$$\|(I-G)(S^n P|_{\mathcal{E}^i L})\|_\infty \le c(\rho_\infty(Y_0, Y_1) + \epsilon)^n\ , \tag{29}$$

for any $\epsilon > 0$. And this, in view of (23), implies that the m-th order derivatives of $S^\infty P$ have Hölder exponent α in $I\!R^2$.

To complete the proof of C^m continuity we use the same method to prove this result for all lower order derivatives of $S^\infty P$. To deal with the k-th order derivatives, for $k < m$, we replace the definition of Δ in (17) by

$$diag(\Delta) = (1, 2, 2, 4, 4, 4, \ldots, 2^k, \ldots, 2^k, 2^{k+1}, \ldots, 2^{k+1}). \tag{30}$$

Also, we redefine of the above projection operator G to be the projection onto the subspace of the $(k+1)(k+2)/2$ right eigenvectors of A corresponding to monomials of degrees $\le k$, namely, $span\{\ Qf|_L\ ,\ \ f = x^i y^j\ ,\ \ 0 \le i+j \le k\}$. In view of the structure of $\widetilde{A}$ and $\widetilde{B}$ in (6),(8), the arguments used for the m-th order derivative can be repeated here, and the claim of the theorem is proved. ∎

7. CONCLUSIONS AND OPEN ISSUES

1. A simple smoothness check. In this paper, we have presented an algorithm for checking the smoothness of quasi-uniform subdivision schemes. It is important to note that the algorithm is simple to apply. It does not require the construction of complicated difference schemes, neither it requires costly eigenvector analysis of the subdivision matrices. The construction of the matrices involved in the algorithm is done by applying the subdivision scheme to specific data. The only eigenvectors needed in the construction correspond to the eigenvalues $1, \frac{1}{2}, \ldots, 2^{-m}$, and are given by (7).

2. The tri-quad example and beyond. The tri-quad mesh serves here as a case study. We use it to demonstrate the construction of the scheme on the boundary between two uniform regions with a different uniform subdivision scheme defined on each. Then we apply the new smoothness check algorithm to the tri-quad scheme. We are not aware of any other method for analyzing such a scheme. The analysis procedure can be directly adapted, or suitably extended, to deal with many other cases of quasi-uniform subdivision (for more examples see [18]). For example, consider a quasi-uniform scheme in $\mathbb{R}^3$, consisting of different uniform schemes on each side of the xy-plane and some special rules near the xy-plane. The smoothness check of such a scheme follows quite the same steps as the algorithm presented in this paper, where in the end one has to estimate the joint spectral radius of four matrices.

3. Necessary and sufficient condition? It is not clear whether the joint spectral radius condition $\rho_\infty(Y_0, Y_1) < 2^{-(m+\alpha)}$ is also necessary for $C^{m+\alpha}$ continuity. It is certainly necessary for the stability of the vector-valued scheme (A_∞, B_∞) and thus for the stability of the non-stationary vector-valued scheme $\{(A_n, B_n)\}$ defined in §4. It turns out that if $\rho_\infty(Y_0, Y_1) > 2^{-(m+\alpha)}$ then the scheme cannot be $C^{m+\alpha}$, but the case of equality is not clear.

4. The analysis of uniform schemes. There are well established analysis tools for uniform multivariate schemes. One approach is via difference schemes ([2, 9, 10]) and the other is in terms of the joint spectral radius of the local subdivision operators ([14, 15, 24]). The method presented here for the analysis of quasi-uniform schemes is related to the second approach. The following result is merely a presentation of the result in [15] in our terminology.

Let us consider a uniform scheme S on $X = \mathbb{Z}^2$, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the shift operators $\mathcal{E}_1 L = \{(i+1, j) | (i,j) \in L\}$, $\mathcal{E}_2 L = \{(i, j+1) | (i,j) \in L\}$. Here we choose $L \subset X$ as the subset of mesh points around the origin such that the values at iteration 1 in L, $\mathcal{E}_1 L$, $\mathcal{E}_2 L$ and $\mathcal{E}_2\mathcal{E}_1 L$, namely $SP|_L$, $SP|_{\mathcal{E}_1 L}$, $SP|_{\mathcal{E}_2 L}$ and $SP|_{\mathcal{E}_2\mathcal{E}_1 L}$, depend only on the initial values in L, $P|_L$. For $(i_1, i_2) \in E \equiv \{(0,0), (1,0), (0,1), (1,1)\}$ let $A^{(i_1,i_2)}$ denote the matrix operator taking the vector of values $P|_L$ to the vector $SP|_{\mathcal{E}_{i_2}\mathcal{E}_{i_1} L}$.

To each point $2^{-n}(i,j) \in 2^{-n}\mathbb{Z}^2$ we relate the vector of data values $v^n_{i,j} = S^n P|_{\mathcal{E}_2^j \mathcal{E}_1^i L}$. The four matrices $\{A^{(i_1,i_2)}\}$ constitute a bivariate vector-valued binary subdivision scheme generating all these vector sequences, namely,

$$v^{n+1}_{2i+i_1, 2j+i_2} = A^{(i_1,i_2)} v^n_{i,j} \ , \quad (i_1, i_2) \in E \ . \tag{31}$$

The benefit in defining such a vector-valued scheme is realized when considered in a special basis. Using the left and right eigenvectors of $A^{(0,0)}$, just as described in the analysis procedure in §3, we form a basis V for the vectors $P|_L$. In this basis, the matrices $A^{(i_1,i_2)}$ take the form

$$\widetilde{A}^{(i_1,i_2)} = \left[\begin{array}{c|c} \Theta^{(i_1,i_2)} & C^{(i_1,i_2)} \\ \hline 0 & Y^{(i_1,i_2)} \end{array} \right], \tag{32}$$

Let the *joint spectral radius* of the four matrices $\{Y^{(i_1,i_2)}\}$ be defined as

$$\rho_\infty(Y^{(0,0)}, Y^{(1,0)}, Y^{(0,1)}, Y^{(1,1)}) =$$

$$\lim_{k \in \mathbb{Z}_+ \setminus 0} \left(\max \left\{ \|Y^{\varepsilon_k} Y^{\varepsilon_{k-1}} \cdots Y^{\varepsilon_1}\|_\infty : \varepsilon_i \in E \right\} \right)^{\frac{1}{k}} . \tag{33}$$

THEOREM 7.1. *Let S be a uniform bivariate binary scheme on $\mathbb{Z}^2$, and assume that S maps Π_m into itself and S^∞ is an injection. Let $\{Y^{(i_1,i_2)}\}$, $(i_1,i_2) \in E$, be defined as above. If*

$$\rho_\infty(Y^{(0,0)}, Y^{(1,0)}, Y^{(0,1)}, Y^{(1,1)}) = 2^{-(m+\alpha)} , \tag{34}$$

$0 < \alpha \le 1$ then S is C^m and the m-th order derivatives of the limit function are Hölder continuous with exponent $\alpha - \epsilon$ for arbitrarily small $\epsilon > 0$.

Proof. The proof relies on checking the decay of the differences of divided differences of the data generated by S, as is done in [2, 9, 10], only without using difference schemes. As in §4, condition (34) implies that the coefficients in the local eigenvector expansion at all dyadic points $2^{-n}\mathbb{Z}^2$, $n \in \mathbb{Z}_+$ are properly bounded. I.e., the coefficients of the eigenvectors with eigenvalue $2^{-(i+j)}$, corresponding to the monomial limit function $x^i y^j$, $i + j \le m$, are $O(2^{-n(i+j)})$, and the coefficients of the other eigenvectors behave as $O(2^{-n(m+\alpha)})$, as $n \to \infty$. We also observe that each vector $v^n_{i,j}$ generated by the vector subdivision (31) represents a subset of values generated by S. Hence, evaluating differences between the elements of $v^n_{i,j}$ is the same as evaluating local differences on $S^n P$ near the point $2^{-n}(i,j)$.

Unlike the quasi-uniform case, using the injectivity assumption, it follows by [18] that the first $(m + 1)(m + 2)/2$ eigenvectors are polynomials (restricted to

$\mathbb{Z}^2$). Moreover, the eigenvector corresponding to the monomial limit function $x^i y^j$, $i + j \le m$, is of the form $(q(x,y) + x^i y^j)|_{\mathbb{Z}^2}$, with $q \in \Pi_{i+j-1}$. Since all divided differences of order $i + j + 1$ of such eigenvector data are zero, we find out that the differences of all divided differences of order m are $O(2^{-n\alpha})$, as $n \to \infty$. Thus, the result follows from the theory in [2, 9, 10]. ∎

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